Compact Knapsack: a Semidefinite Approach

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Problem statement

Items labelled $i \in \{1, \dots, n\}$, with costs c_i and weights w_i, $q \in \mathbf{R}_+$.

- **Knapsack**: find a selection $S \subseteq \{1, \dots, n\}$ that minimizes the total cost and verifies
$$\sum_{i \in S} w_i \geqslant q$$
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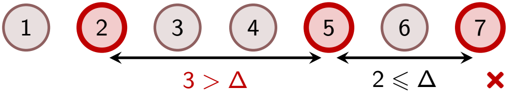

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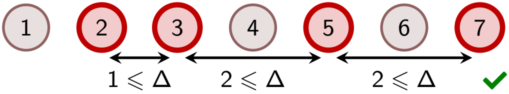

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Strengthening coefficient

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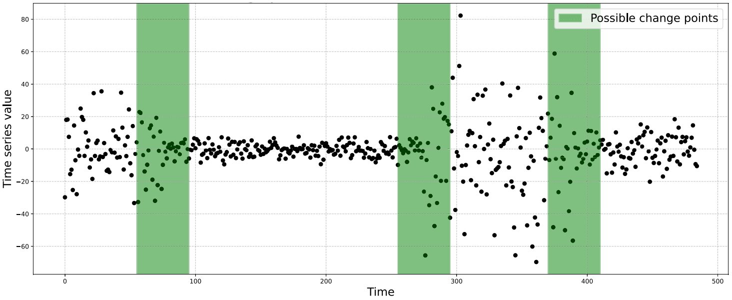

A time series with its possible change points in variance **[*Santini and Malaguti, 2024*]**

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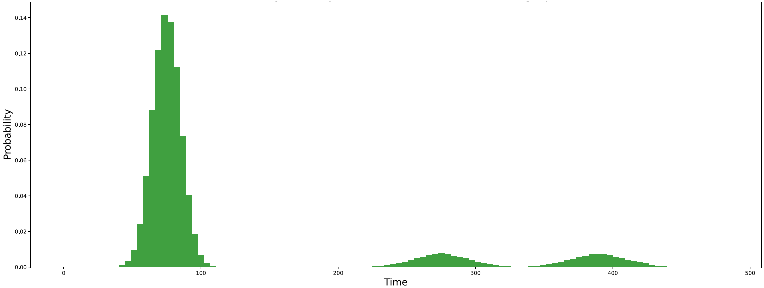

Probabilities associated with each time point
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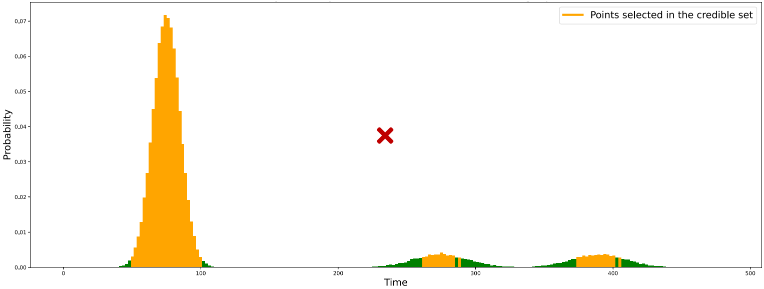

Credible set relative to the first change point **[*Santini and Malaguti, 2024*]**

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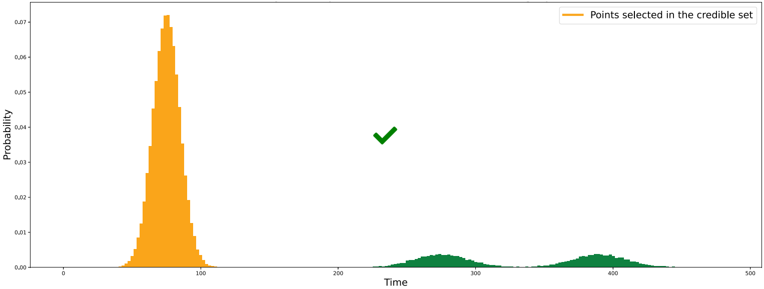

Credible set relative to the first change point with the compactness constraint **[*Santini and Malaguti, 2024*]**

Modelling as an SDP

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Theorem (Classical, see e.g. ***De Meijer and Sotirov, 2024***)

Let $\overline{X} = \begin{pmatrix} 1 & \operatorname{diag}(X)^\top \\ \operatorname{diag}(X) & X \end{pmatrix} \succeq 0$, $X \neq 0$. The following are equivalent:

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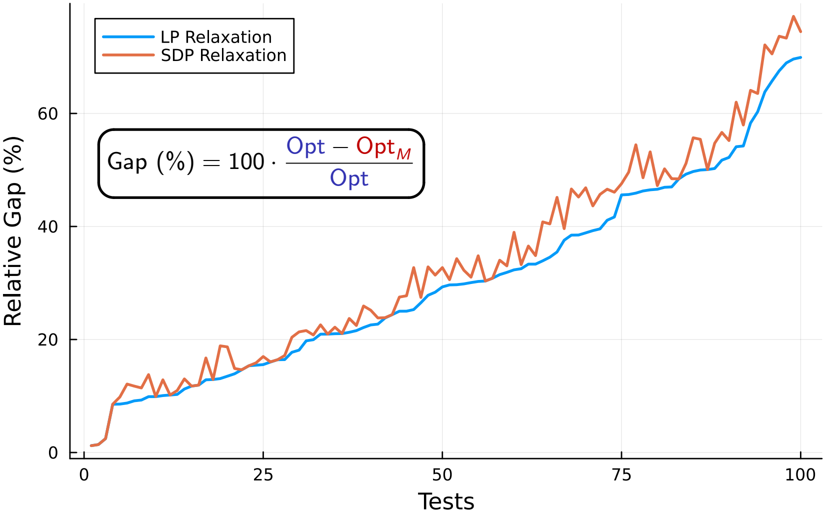

Opt is the optimal integer solution and Opt_M is the optimal solution returned by model M; here for the linear (—) and semidefinite (—) relaxations

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$$(1 - x_i)(1 - x_j) \geqslant 0$$
$$x_i x_j \geqslant x_i + x_j - 1$$

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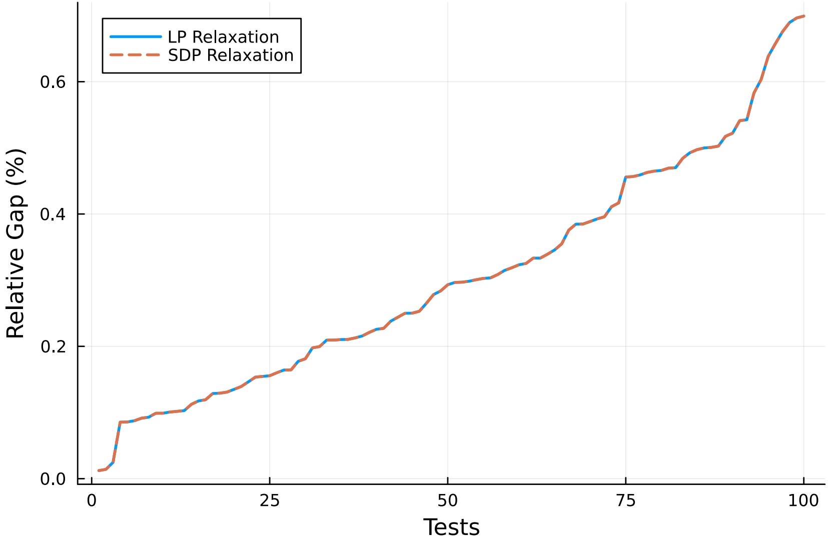

With the $X_{ij} \geqslant X_{ii} + X_{jj} - 1$ inequalities

If x is a solution of

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Conjecture

Above X is a solution of $(\text{min-KPC})_{\text{SDP}}$.
In particular,

$$\text{Opt}\left((\text{min-KPC})_{\text{SDP}}\right) = \text{Opt}\left((\text{min-KPC})_{\text{LP}}\right) \leqslant \text{Opt}\left((\text{min-KPC})_{\text{int}}\right).$$

Since $x \in \{0,1\}^n$, we deduce the following constraints:

For all $i, j, k \in [\![n]\!]$:

$$
\begin{aligned}
X_{ij} &\geqslant 0 \\
X_{ii} &\geqslant X_{ij} \\
X_{ij} &\geqslant X_{ii} + X_{jj} - 1 \\
X_{kk} + X_{ij} &\geqslant X_{ik} + X_{jk} \\
X_{ik} + X_{jk} + X_{ij} + 1 &\geqslant X_{ii} + X_{jj} + X_{kk}
\end{aligned}
$$

Cauchy-Schwarz inequality on $\mathrm{tr}\left(\mathrm{Diag}(w)X\right)$ yields:

$$
\sum_{i=1}^{n} w_i^2 X_{ii} + 2 \sum_{1 \leqslant i < k \leqslant n} w_i w_k X_{ik} \leqslant \left(\sum_{i=1}^{n} w_i^2\right)\left(\sum_{1 \leqslant i,k \leqslant n} X_{ik}\right)
$$

With the added quadratic constraints

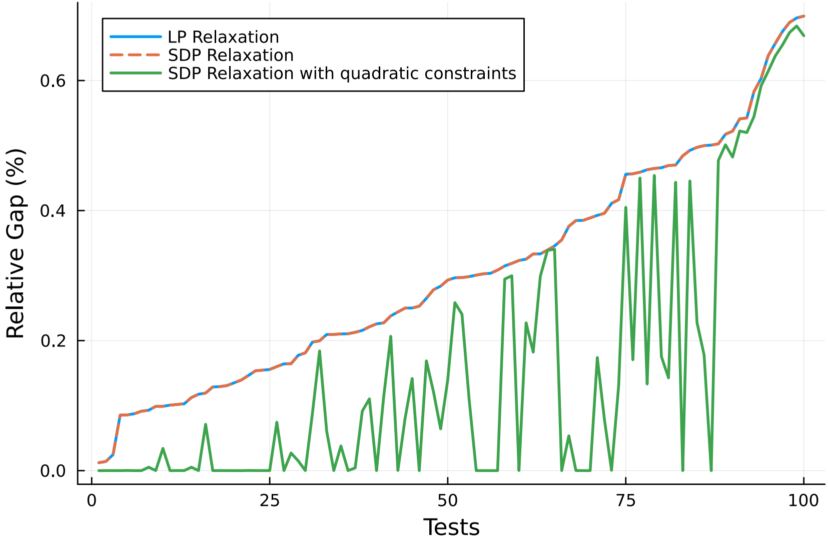

Figure 2: relative gap for the model with the semidefinite relaxation when

Maximal insufficient subset constraints

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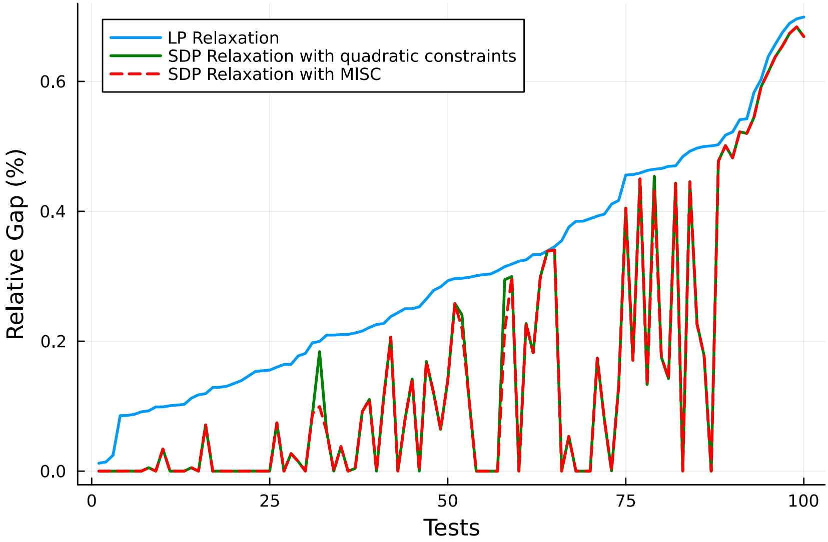
Greedy algorithm to compute maximal insufficients subsets

$S \leftarrow$ random sufficient subset
while $\sum_{i \in S} w_i \geqslant q$ **do**
  Remove the heaviest object in S.
end while
return S

With MISC cuts



Model with a randomly generated (MISC) (- -) in comparison with the linear relaxation (—) and the semidefinite relaxation (—)

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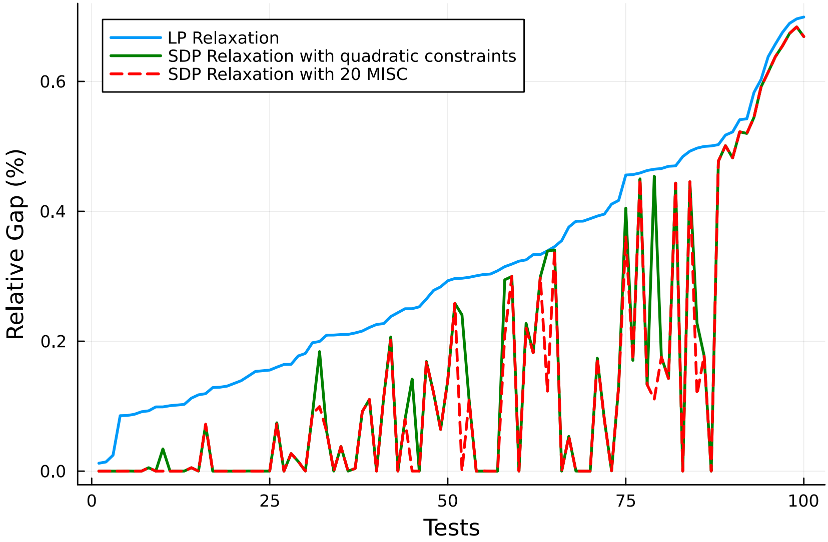

Model with a several independently randomly generated (MISC) (- -) in comparison with the linear relaxation (—) and the semidefinite relaxation (—)

Recalls: reduced cost fixing

Consider a linear problem (P)

$$z_P = \min \left\{ c^\top x \,\middle|\, Ax \geqslant b, x \geqslant 0 \right\}$$

and a known upper bound for (P), $z_{\mathrm{UB}} \geqslant z_P$.

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Linear relaxation $(\text{min-KPC})_{\text{LP}}$

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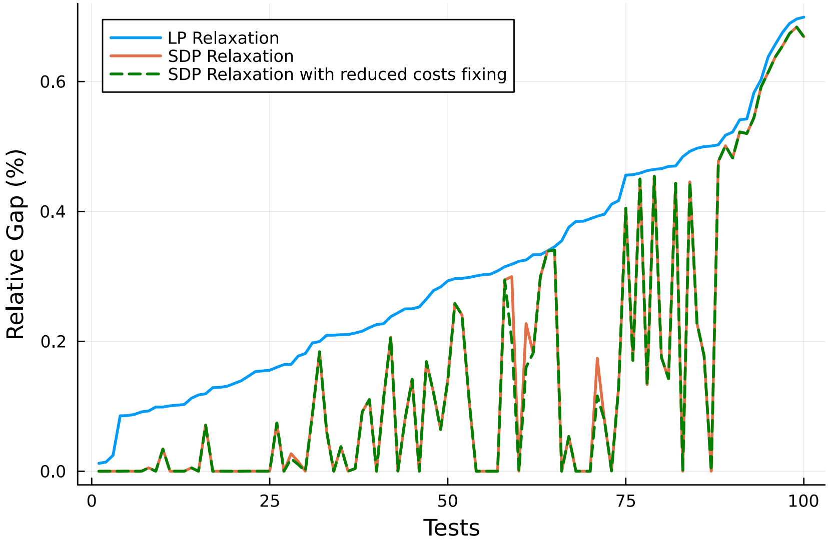

Model where some variable are fixed with a pre-solve (- -) in comparison with the linear relaxation (—) and the semidefinite relaxation (—)

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- Semidefinite relaxation effectively improves the bounds on this combinatorial problem when we tighten the model with quadratic constraints.
- The linear relaxation can be used to presolve our model with a reduced cost fixing heuristic, and to generate a maximal insufficient subset that separates an optimal fractional point.

Thank you for your attention!

References

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Appendix - PSD matrices

Definition (Positive semidefinite matrix)

A symmetric matrix $X \in \mathrm{M}_n(\mathbf{R})$ is *positive semidefinite* if for all $v \in \mathbf{R}^n$, $v^\top X v \geqslant 0$. We write $X \succeq 0$.

Properties

- $X \succeq 0 \iff X = \sum_{i=1}^{r} \lambda_i x_i x_i^\top$ with $\lambda_i \geqslant 0$ and $x_i \in \mathbf{R}^n$.
- $X \succeq 0 \iff$ all prinicpal minors of X are nonnegative.

Proposition (Schur complement's lemma)

Let X be the symmetric matrix defined by

$$X = \begin{pmatrix} A & B^\top \\ B & C \end{pmatrix}$$

with A invertible. Then $X \succeq 0$ if and only if $C - BA^{-1}B^\top \succeq 0$.