Compact Knapsack: a Semidefinite Approach

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Problem statement

Items $i \in \{1, \ldots, n\}$, with costs c_i and weights w_i.

- **min-Knapsack**: find a selection $S \subseteq \{1, \ldots, n\}$ that minimizes the total cost and verifies
$$\sum_{i \in S} w_i \geq q$$
- **Compactness**: **[*Santini and Malaguti, 2024*]**
S contains no gap that exceed Δ.

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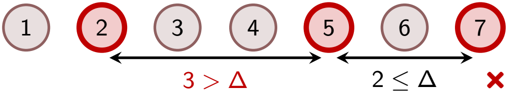

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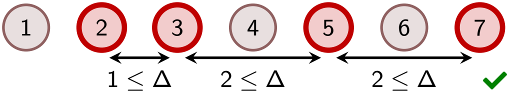

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$$\left[\begin{array}{rl} \text{minimize} & c^\top x \\ \text{subject to} & w^\top x \geq q \\ & \forall i, j \in [\![n]\!],\ j - i > \Delta, \quad x_i + x_j - 1 \leq \displaystyle\sum_{k=i+1}^{j-1} x_k \\ & x \in \{0, 1\}^n \end{array}\right.$$

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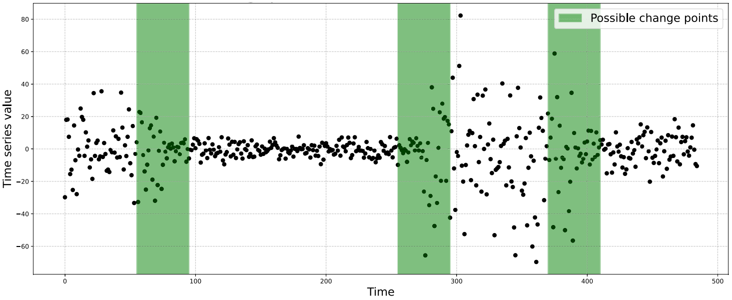
Strengthening coefficient

$$\left[\begin{array}{lll} \text{minimize} & c^\top x & \\ \text{subject to} & w^\top x \geq q & \\ & \forall i, j \in [\![n]\!],\ j - i > \Delta, & \left\lfloor \dfrac{j-i-1}{\Delta} \right\rfloor (x_i + x_j - 1) \leq \sum_{k=i+1}^{j-1} x_k \\ & x \in \{0,1\}^n & \end{array}\right.$$

Given a time series $\{y_1, \ldots, y_n\}$, how to detect changes?

Method **[*Cappello and Padilla, 2022*]** focuses **most probable** change point:
$\rightarrow$ Each point gets a probability of being the change point.



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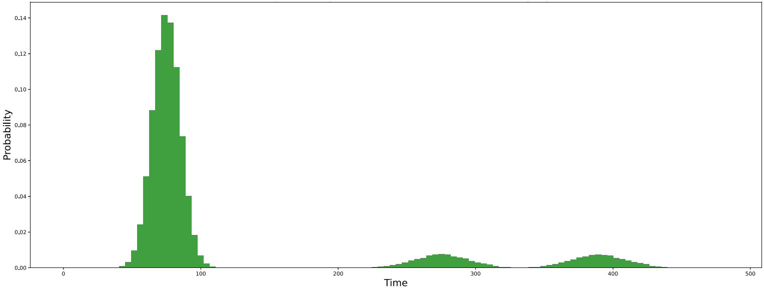

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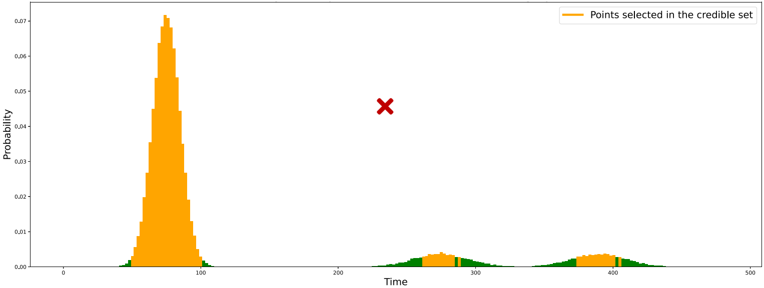

Credible set relative to the first change point

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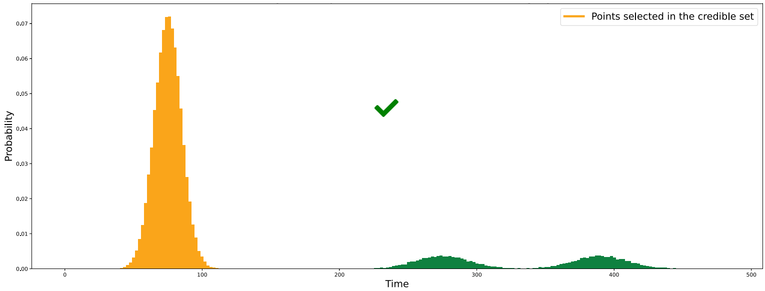

Credible set relative to the first change point with the compactness constraint

$$
\left[\begin{array}{rl}
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\text{subject to} & w^\top x \geq q \\
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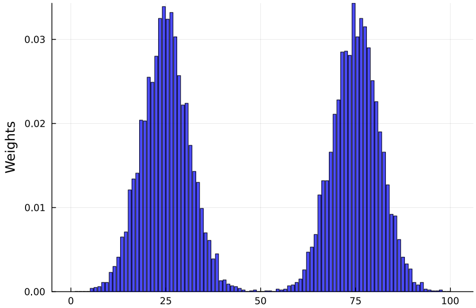

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Substitute $X = xx^\top$. Then $X_{ij} = x_i x_j$ and $X_{ii} = x_i^2 = x_i$.

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\left[\begin{array}{rl}
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& X \text{ has coefficients in } \{0,1\} \\
& \operatorname{rank}(X) = 1 \\
& X \succeq 0
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Theorem (Classical, see e.g. ***De Meijer and Sotirov, 2024***)

Let $\overline{X} = \begin{pmatrix} 1 & \operatorname{diag}(X)^\top \\ \operatorname{diag}(X) & X \end{pmatrix} \succeq 0$, $X \neq 0$. The following are equivalent:

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Relaxation!

$$\left[\begin{array}{rl} \text{minimize} & c^\top \operatorname{diag}(X) \\ \text{subject to} & w^\top \operatorname{diag}(X) \geq q \\ & \forall i,j \in [\![n]\!],\ j-i > \Delta, \quad \left\lfloor \dfrac{j-i-1}{\Delta} \right\rfloor X_{ij} \leq \displaystyle\sum_{k=i+1}^{j-1} X_{kk} \\ & \begin{pmatrix} 1 & \operatorname{diag}(X)^\top \\ \operatorname{diag}(X) & X \end{pmatrix} \succeq 0 \\ & \cancel{\operatorname{rank}(X) = 1} \end{array}\right.$$

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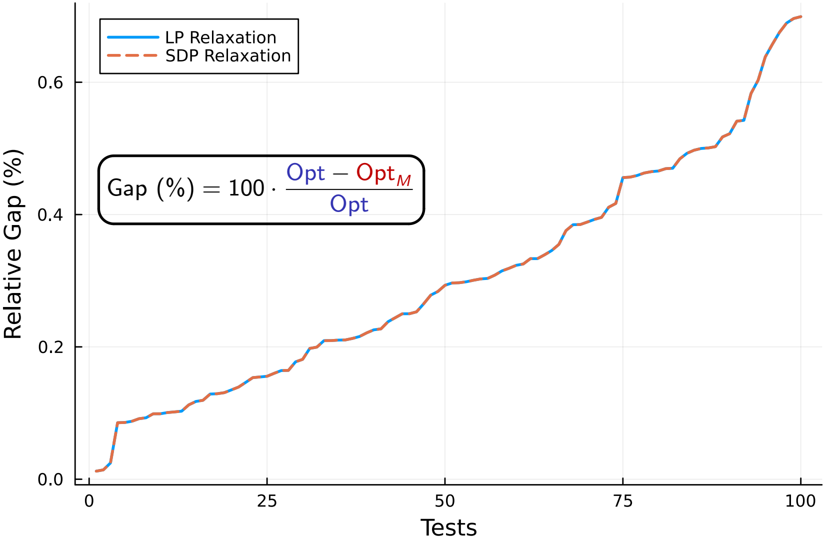
SDP versus LP 😭



Opt is the optimal integer solution and Opt_M is the optimal solution returned by model M; here for the linear (—) and semidefinite (—) relaxations

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 & \operatorname{tr}\left(\operatorname{Diag}(w)^\top X\right)^2 \leq \|\operatorname{Diag}(w)\|^2 \cdot \|X\|^2
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Modelling as an SDP

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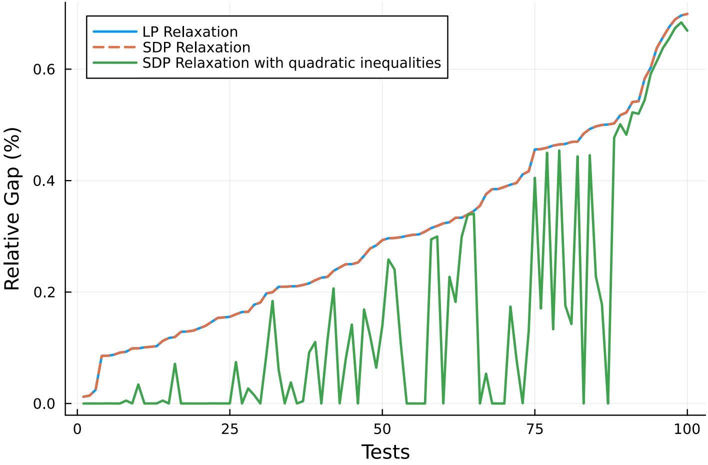
Quadratic Inequalities

With the added quadratic inequalities



Relative gap for the semidefinite relaxation without (- - -) *vs.* with (——) the quadratic inequalities

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Modelling as a penalized SDP

Let $(\lambda_{ij})_{1 \le i < j \le n} \in \mathbf{R}^{n(n-1)/2}$

$$
\left[\begin{array}{ll}
\text{minimize} & c^\top \operatorname{diag}(X) + \sum_{1 \le i < j \le n} \lambda_{ij} \left(\left\lfloor \frac{j-i-1}{\Delta} \right\rfloor X_{ij} - \sum_{k=i+1}^{j-1} X_{kk} \right) \\
\text{subject to} & w^\top \operatorname{diag}(X) \ge q \\
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Let $x \in [0,1]^n$ be the solution vector for the min-KPC.

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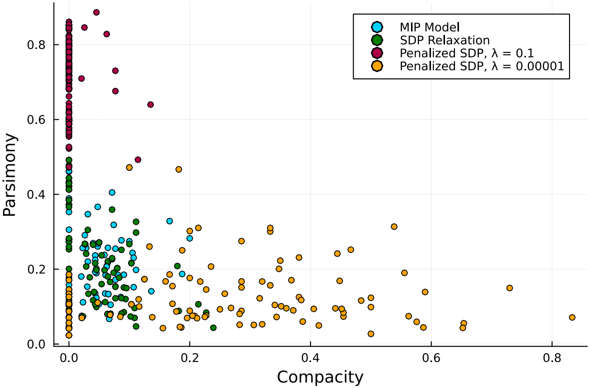
- **Fractionnality**

$$\frac{2}{\sqrt{n}} \cdot \left\| x - \left\lfloor x + \frac{1}{2} \right\rfloor \right\|_2$$

Benchmark: 100 instances; 4 models.



Measuring the quality of a solution

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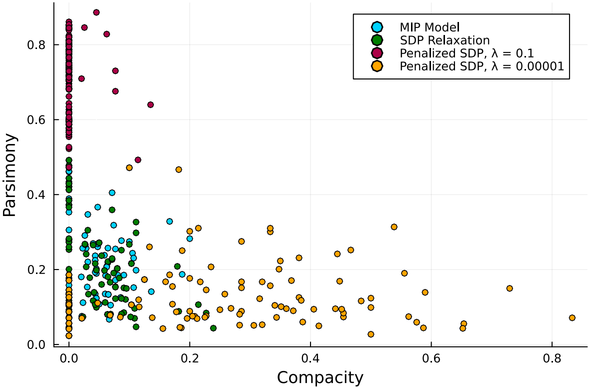

Model	Average fractionnality
SDP Relaxation	0.1904
Penalized SDP with $\lambda = 0.1$	0.0028
Penalized SDP with $\lambda = 0.00001$	0.0429

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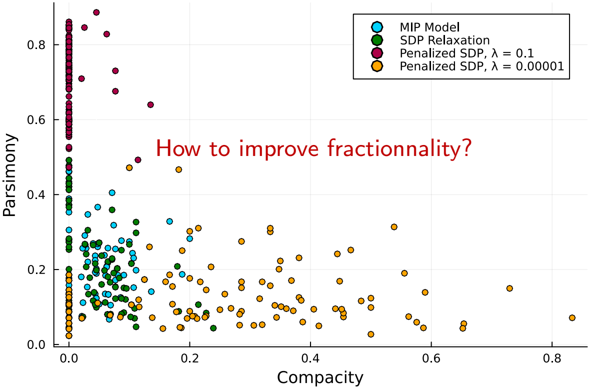

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$$\sum_{i \notin S} X_{ii} \geq 1 \tag{MISC}$$

Question

Given X^* a fractionnal point, how to find S maximal insufficient subset such that (MISC) separates X^*?

MISC separation problem

$$\text{(MISC) separates } X^* \quad \Leftrightarrow \quad S \text{ M.I.S. such that } \sum_{i \notin S} X^*_{ii} < 1$$

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$$\text{(MISC) separates } X^* \quad \Leftrightarrow \quad S \text{ M.I.S. such that } \sum_{i=1}^{n} (1 - \mathbb{1}_S(i))\, X^*_{ii} < 1$$

$$\left[\begin{array}{rl} \text{minimize} & \displaystyle\sum_{i=1}^{n} (1 - \alpha_i)\, X^*_{ii} \\ \text{subject to} & \displaystyle\sum_{i=1}^{n} w_i \alpha_i < q \\ & \alpha \in \{0,1\}^n \end{array}\right.$$

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$S \leftarrow \{i \in [\![n]\!] \mid \alpha^*_i = 1\}$
while $\sum_{i \in S} w_i < q$ **do**
  Add $j \in [\![n]\!] \setminus S$ of minimal weight to S
end while
Remove the last item that was added
return S

MISC separation problem

$$\text{(MISC) separates } X^* \quad \Leftrightarrow \quad S \text{ M.I.S. such that } \sum_{i=1}^{n} (1 - \mathbb{1}_S(i))\, X^*_{ii} < 1$$

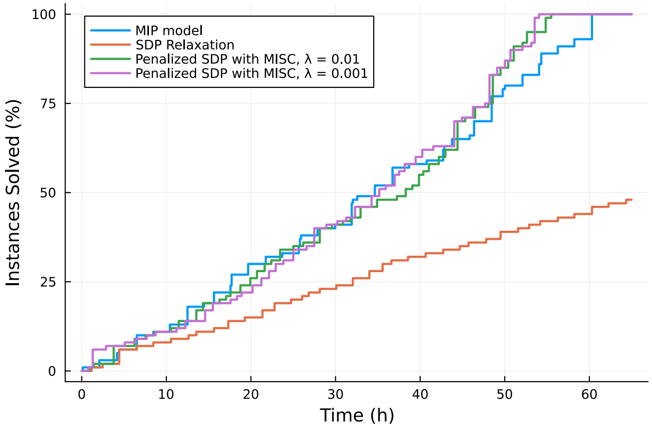
$$\left[\begin{array}{ll} \text{minimize} & \displaystyle\sum_{i=1}^{n} (1-\alpha_i)\, X^*_{ii} \\ \text{subject to} & \displaystyle\sum_{i=1}^{n} w_i \alpha_i < q \\ & \alpha \in \{0,1\}^n \end{array}\right.$$

$S \leftarrow \{i \in [\![n]\!] \mid \alpha^*_i = 1\}$
while $\sum_{i \in S} w_i < q$ **do**
  Add $j \in [\![n]\!] \setminus S$ of minimal weight to S
end while
Remove the last item that was added
return S

Proposition (adapted from knapsack litterature)

- S is a maximal insufficient subset
- If $\sum_{i=1}^{n} (1-\alpha^*_i)\, X^*_{ii} < 1$, then $\sum_{i \notin S} X^*_{ii} < 1$
- If $\sum_{i=1}^{n} (1-\alpha^*_i)\, X^*_{ii} \geq 1$, then no (MISC) separate X^*_{ii}.

Benchmark: 100 instances with $n \in \{200, \ldots, 400\}$.



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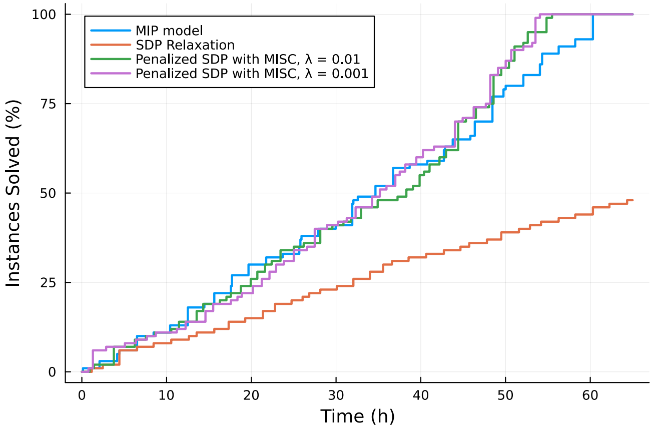

Model	Average fractionnality
SDP Relaxation	0.2509
Penalized SDP with $\lambda = 0.01$	1.3107e-5
Penalized SDP with $\lambda = 0.001$	6.7288e-6

Conclusion

- Compactness constraint brings a new layer of difficulties to the standard knapsack problem.

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- The penalized version provides high-quality heuristics and tight bounds for the problem.
- The penalized version allows tuning the model to the appropriate balance between parsimony and compacity.
- Future work is needed to strenghen the model and to improve numerical performance.

References

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Definition (Positive semidefinite matrix)

A symmetric matrix $X \in \mathrm{M}_n(\mathbf{R})$ is *positive semidefinite* if for all $v \in \mathbf{R}^n$, $v^\top X v \geq 0$. We write $X \succeq 0$.

Properties

- $X \succeq 0 \iff X = \sum_{i=1}^{r} \lambda_i x_i x_i^\top$ with $\lambda_i \geq 0$ and $x_i \in \mathbf{R}^n$.
- $X \succeq 0 \iff$ all prinicpal minors of X are nonnegative.

Proposition (Schur complement's lemma)

Let X be the symmetric matrix defined by

$$X = \begin{pmatrix} A & B^\top \\ B & C \end{pmatrix}$$

with A invertible. Then $X \succeq 0$ if and only if $C - BA^{-1}B^\top \succeq 0$.

Appendix - Counterexample: SDP vs LP

With 10 items, costs $c_1 = 1$ for all $i \in [\![10]\!]$, $w_1 = w_{10} = 11$, $w_j = 1$ for all $j \neq 1, 10$, $q = 22$ and $\Delta = 2$.

$$\left[\begin{array}{rl} \text{minimize} & x_1 + \cdots + x_{10} \\ \text{subject to} & 11x_1 + x_2 + \cdots + x_9 + 11x_{10} \geq 22 \\ & \forall i,j \in [\![10]\!], j-i > 2, \quad \left\lfloor \dfrac{j-i-1}{2} \right\rfloor (x_i + x_j - 1) \leq \displaystyle\sum_{k=i+1}^{j-1} x_k \\ & x \in [0,1]^{10} \end{array}\right.$$

Set $\mathbf{X} = (x^*_{\mathbf{LP}})^\top x^*_{\mathbf{LP}}$. Then $\mathbf{X}$ is not a solution of

$$\left[\begin{array}{rl} \text{minimize} & X_{1\,1} + \cdots + X_{10\,10} \\ \text{subject to} & 11X_{1\,1} + X_{2\,2} + \cdots + X_{9\,9} + 11X_{10\,10} \geq 22 \\ & \forall i,j \in [\![10]\!], j-i > 2, \quad \left\lfloor \dfrac{j-i-1}{2} \right\rfloor X_{ij} \leq \displaystyle\sum_{k=i+1}^{j-1} X_{kk} \\ & \begin{pmatrix} 1 & \operatorname{diag}(X)^\top \\ \operatorname{diag}(X) & X \end{pmatrix} \succeq 0. \end{array}\right.$$

Appendix - Counterexample: SDP vs LP

$$\left[\begin{array}{rl} \text{minimize} & x_1 + \cdots + x_{10} \\ \text{subject to} & 11x_1 + x_2 + \cdots + x_9 + 11x_{10} \geq 22 \\ & \forall i,j \in [\![10]\!], j-i > 2, \quad \left\lfloor \dfrac{j-i-1}{2} \right\rfloor (x_i + x_j - 1) \leq \displaystyle\sum_{k=i+1}^{j-1} x_k \\ & x \in [0,1]^{10} \end{array}\right.$$

$$x^*_{\mathbf{LP}} = \left(1\,,\ \frac{3}{4}\,,\ \frac{119}{180}\,,\ 0\,,\ \frac{17}{135}\,,\ \frac{251}{540}\,,\ 0\,,\ \frac{107}{540}\,,\ \frac{11}{15}\,,\ \frac{11}{15}\right)$$

$$\left[\begin{array}{rl} \text{minimize} & X_{1\,1} + \cdots + X_{10\,10} \\ \text{subject to} & 11X_{1\,1} + X_{2\,2} + \cdots + X_{9\,9} + 11X_{10\,10} \geq 22 \\ & \forall i,j \in [\![10]\!], j-i > 2, \quad \left\lfloor \dfrac{j-i-1}{2} \right\rfloor X_{ij} \leq \displaystyle\sum_{k=i+1}^{j-1} X_{kk} \\ & \begin{pmatrix} 1 & \operatorname{diag}(X)^\top \\ \operatorname{diag}(X) & X \end{pmatrix} \succeq 0. \end{array}\right.$$

X does not verify the $(2,9)$-compacity constraint:

$$\left\lfloor \frac{9-2-1}{2} \right\rfloor \mathbf{X}_{2\,9} = \boxed{\frac{33}{20} > \frac{29}{20}} = \sum_{k=3}^{8} \mathbf{X}_{kk}$$

$$\operatorname{Opt}(\mathrm{SDP}) \approx 4.42 < 4.66 \approx \operatorname{Opt}(\mathrm{LP})$$

Appendix - How to generate instances?

Focus on "hard" instances from **[*Santini and Malaguti, 2024*]**: **TwoPeaks** instances.

- Choose peaks location $\lambda_1 \rightsquigarrow \mathcal{N}(n/3, n/6)$, $\lambda_2 \rightsquigarrow \mathcal{N}(2n/3, n/6)$.

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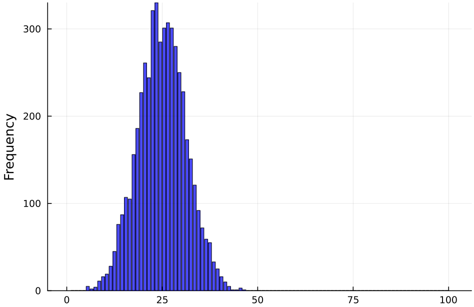
- Histogramm 5000 samples for each peak

$$1^{\text{st}} \text{ peak} \quad w_1 \rightsquigarrow \mathcal{N}(\lambda_1, n/2k) \qquad\qquad 2^{\text{nd}} \text{ peak} \quad w_2 \rightsquigarrow \mathcal{N}(\lambda_2, n/2k)$$

where $k \in \{8, 16, 32\}$.



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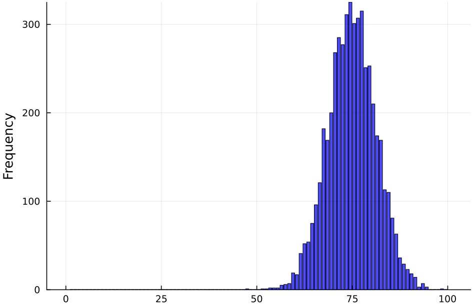

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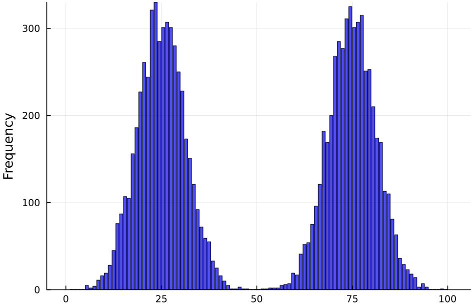
  1^{st} peak $\quad w_1 \rightsquigarrow \mathcal{N}(\lambda_1, n/2k)$ $\qquad$ 2^{nd} peak $\quad w_2 \rightsquigarrow \mathcal{N}(\lambda_2, n/2k)$

  where $k \in \{8, 16, 32\}$.
- Set $w \leftarrow w_1 + w_2$ and normalize.



Appendix - Penalized versions

For $\lambda \in \mathbf{R}^{n(n+1)/2}$:

$$\left[\begin{array}{rl} \text{minimize} & c^\top \operatorname{diag}(X) + \varphi(\lambda, X) \\ \text{subject to} & w^\top \operatorname{diag}(X) \geq q \\ & \begin{pmatrix} 1 & \operatorname{diag}(X)^\top \\ \operatorname{diag}(X) & X \end{pmatrix} \succeq 0 \\ & \text{Quadratic constraints} \end{array}\right.$$

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Pay&Reward

$$\text{minimize} \qquad c^\top \operatorname{diag}(X) + \sum_{i<j} \lambda_{ij} \left(\left\lfloor \frac{j-i-1}{\Delta} \right\rfloor X_{ij} - \sum_{k=i+1}^{j-1} X_{kk} \right)$$

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MaxGap

$$\begin{array}{rl} \text{minimize} & c^\top \operatorname{diag}(X) + \tau \\ \text{subject to} & \tau \geq \lambda_{ij} \left(\left\lfloor \dfrac{j-i-1}{\Delta} \right\rfloor X_{ij} - \sum_{k=i+1}^{j-1} X_{kk} \right) \end{array}$$

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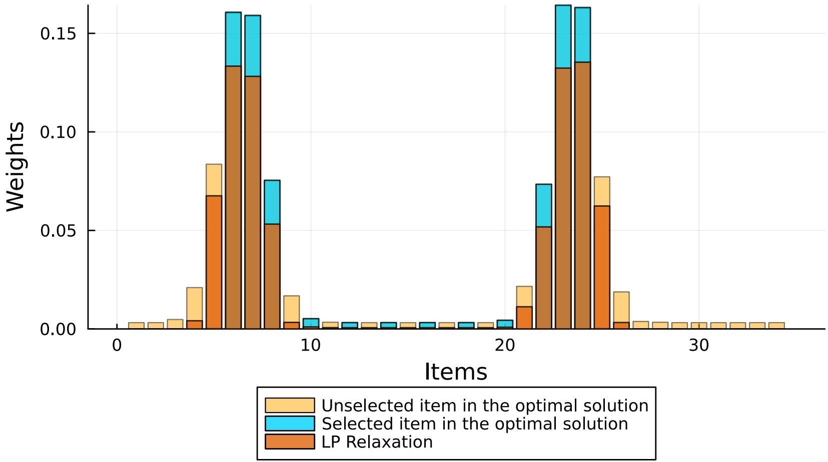
PayEachGap

$$\begin{array}{rl} \text{minimize} & c^\top \operatorname{diag}(X) + \sum_{i<j} \lambda_{ij} \left\lfloor \dfrac{j-i-1}{\Delta} \right\rfloor X_{ij} \\ \text{subject to} & \left\lfloor \dfrac{j-i-1}{\Delta} \right\rfloor X_{ij} \leq \sum_{k=i+1}^{j-1} X_{kk} \end{array}$$

Appendix - Effets of the different penalizations



selected items for the linear relaxation, with $\Delta = 2$

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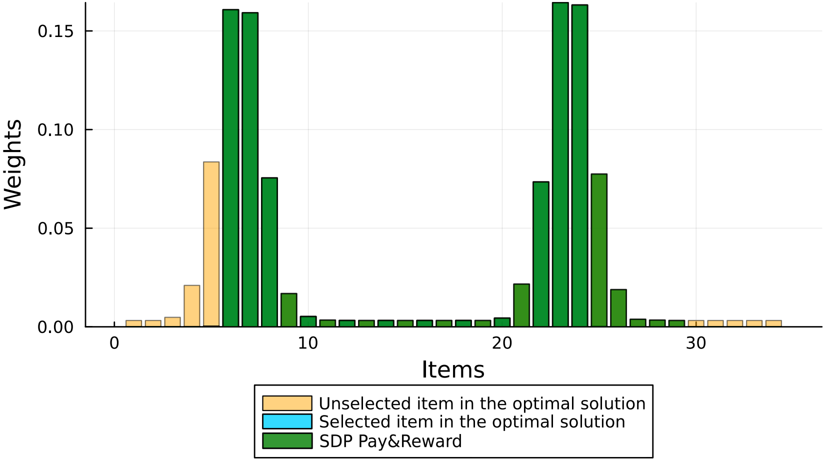
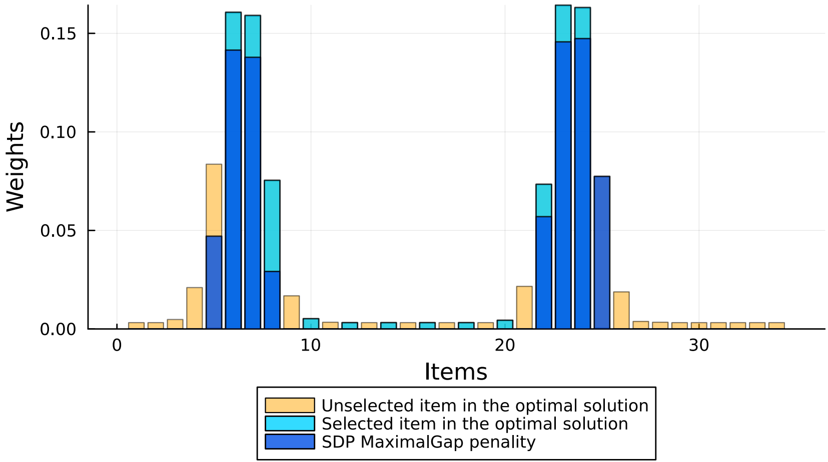


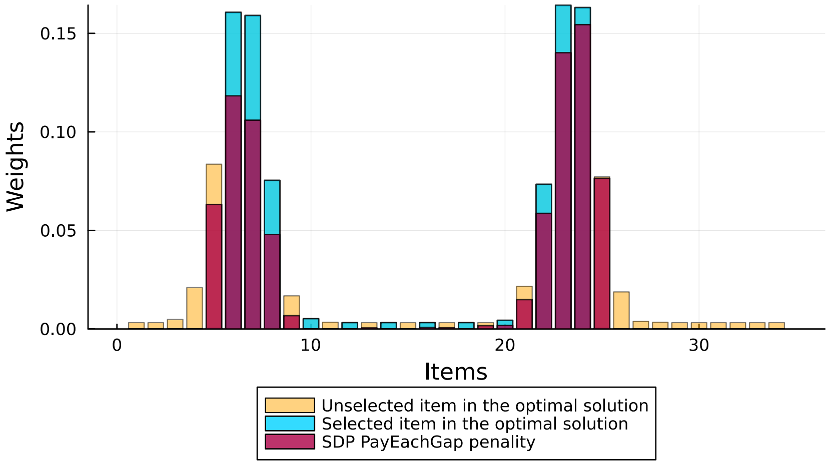
selected items for the Pay&Reward penalization, with $\Delta = 2$

Appendix - Effets of the different penalizations



selected items for the MaxGap penalization, with $\Delta = 2$

Appendix - Effets of the different penalizations



selected items for the PayEachGap penalization, with $\Delta = 2$