Knapsack with compactness: a semidefinite approach

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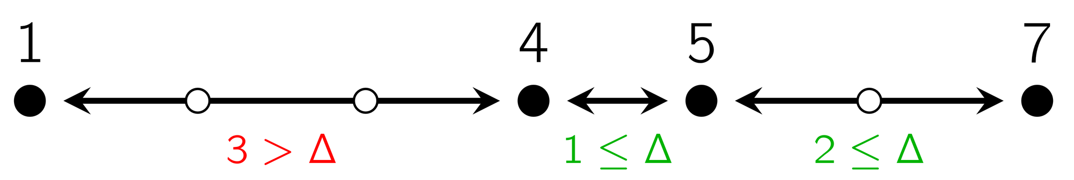
Problem statement

Knapsack with compactness constraints: extension of the classical minimization knapsack: n items $i \in [\![n]\!]$ with costs c_i and weights w_i.

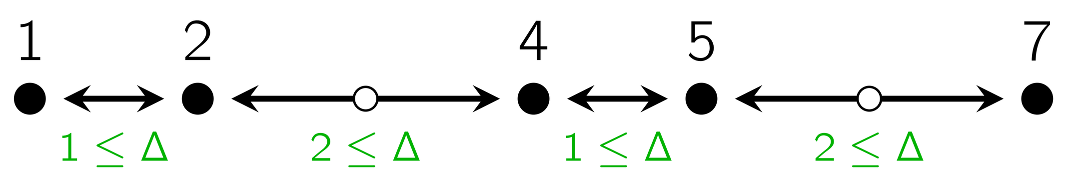
- **Goal**: find $S \subseteq [\![n]\!]$ minimizing the total cost while ensuring that the total weight exceeds a given threshold q.
- **Compactness**: the selected items must remain grouped together.

There must not have gaps of more than $\Delta \geq 1$ in between two consecutive selected items. [3]

For example, with $\Delta = 2$:



(Fig 1: *non-compact* selection)



(Fig 2: *compact* selection)

The problem has applications in statistics, *e.g.* **detecting changes in time series** [1].

Models

- **Imput**: costs $c \in \mathbf{R}^n_+$, weights $w \in \mathbf{R}^n_+$, threshold $q \in \mathbf{R}_+$
- **Variables**: $\mathbf{x}_i$ binary, ($\mathbf{x}_i = 1 \Leftrightarrow$ item i is selected).

$$\left[\begin{array}{ll} \text{minimize} & c^\top \mathbf{x} \\ \text{subject to} & w^\top \mathbf{x} \geq q \\ & \textbf{compactness constraints} \\ & \mathbf{x} \in \{0,1\}^n \end{array}\right.$$

Two expressions for the compactness constraints: $i, j \in [\![n]\!]$ with $j - i > \Delta$:

$$\mathbf{x}_i + \mathbf{x}_j - 1 \leq \sum_{k=i+1}^{j-1} \mathbf{x}_k \qquad (1a)$$

(**Linear** *compactness constraint - [3] approach*)

$$\mathbf{x}_i \mathbf{x}_j \leq \sum_{k=i+1}^{j-1} \mathbf{x}_k. \qquad (1b)$$

(**Quadratic** *compactness constraint - our approach*)

Semidefinite relaxation: with the matrix $\mathbf{X}$ as variable, $c^\top \mathrm{diag}(\mathbf{X})$ for the objective function, the knapsack constraint $w^\top \mathrm{diag}(\mathbf{X}) \geq q$ and the constraints:

$$\underbrace{\left\lfloor \frac{j-i-1}{\Delta} \right\rfloor}_{\text{Strenghening coefficient}} \mathbf{X}_{ij} \leq \sum_{k=i+1}^{j-1} \mathbf{X}_{kk} \qquad (1b)_{\text{SDP}}$$

$$\begin{pmatrix} 1 & \mathrm{diag}(\mathbf{X})^\top \\ \mathrm{diag}(\mathbf{X}) & \mathbf{X} \end{pmatrix} \succeq 0. \qquad (2)$$

Our approach: penalized model

The compactness is enforced in the objective rather than in the constraints: for a penalization parameter $\lambda \in \mathbf{R}_+$, the objective function becomes

$$c^\top \mathrm{diag}(\mathbf{X}) + \lambda \sum_{\substack{1 \leq i \leq j \leq n \\ j-i > \Delta}} \left(\left\lfloor \frac{j-i-1}{\Delta} \right\rfloor \mathbf{X}_{ij} - \sum_{k=i+1}^{j-1} \mathbf{X}_{kk} \right).$$

Strengthening inequalities

Valid quadratic inequality, *linearize* and *lift* to the matrix space:

$$\mathbf{x}_i \mathbf{x}_j \leftrightarrow \mathbf{X}_{ij} \qquad \mathbf{x}_i^2 \leftrightarrow \mathbf{X}_{ii}$$
$$\mathbf{X}_{ij} \leftrightarrow \mathbf{X}_{ji} \qquad \mathbf{X}_{ii}^2 \leftrightarrow \mathbf{X}_{ii}$$

- **Sherali-Adams inequalities**: $\forall i, j \in [\![n]\!]$

$$\mathbf{x}_i \mathbf{x}_j \geq 0 \quad \mathbf{x}_i(1 - \mathbf{x}_j) \geq 0 \quad (1 - \mathbf{x}_i)(1 - \mathbf{x}_j) \geq 0$$

- **Quadratic knapsack constraints**: $\forall i \in [\![n]\!]$

$$w^\top \mathbf{x} \mathbf{x}_i \geq q \mathbf{x}_i \qquad w^\top \mathbf{x}(1 - \mathbf{x}_i) \geq q(1 - \mathbf{x}_i)$$

$\rightarrow$ Strengthends SDP models.

Maximal Insufficient Subsets

Maximal Insufficient Subset: $S \subseteq [\![n]\!]$

$$\forall j \in [\![n]\!] \setminus S \qquad q - w_j \leq \sum_{i \in S} w_i < q.$$

Valid inequality:
$$\sum_{i \notin S} \mathbf{X}_{ii} \geq 1. \qquad \text{(MISC)}$$

Let X^*: optimal solution of SDP model. Then

$$\left[\begin{array}{ll} \text{minimize} & \mathrm{diag}(X^*)^\top (1 - \mathbf{a}) \\ \text{subject to} & w^\top \mathbf{a} \leq q - \varepsilon \\ & \mathbf{a} \in \{0,1\}^n \end{array}\right. \qquad (3)$$

Separation algorithm

For a SDP model M:

$X^* \leftarrow \mathrm{argmin}(M)$; solve (3)
if Opt(3) ≥ 1 **then** ($\nexists$ (MISC) separating $\mathbf{X}^*$)
  return X^*
else
  $S \leftarrow$ correct $\left(\overline{S} = \{i \text{ s.t. } \mathbf{a}_i^* = 1\}\right)$ ($\star$)
  return argmin (M with (MISC))
end if

($\star$) **while** $\sum_{i \in \overline{S}} w_i < q$
  $\overline{S} \leftarrow \overline{S} \cup \{\mathrm{argmin}_j\, w_j\}$
  while end; remove the last item added

Metrics

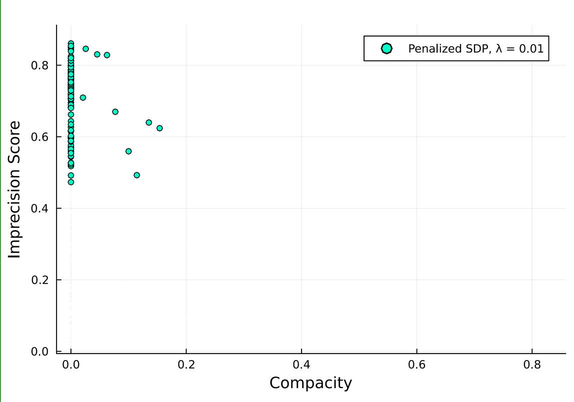
Imprecision
$$\mathbf{imp}(\mathbf{x}) = \frac{c^\top \mathbf{x}}{\|c\|_1}$$

Compacity
$$\mathbf{comp}(S) = \frac{1}{n} \max_{i,j \in S} \left\{ j - i - 1 \;\middle|\; \begin{array}{c} i, j \\ \text{consecutive} \end{array} \right\}$$

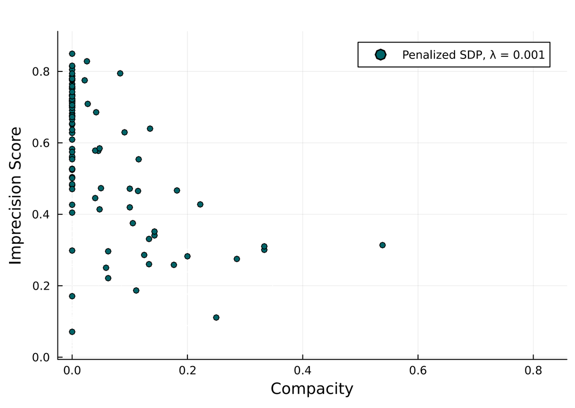
Fractionnality
$$\mathbf{frac}(\mathbf{x}) = \frac{2}{\sqrt{n}} \left\| \mathbf{x} - \left\lfloor \mathbf{x} + \frac{1}{2} \cdot \mathbf{1} \right\rfloor \right\|_2$$

λ	Average fractionnality (**frac**)	
	Penalized SDP	Separation Algorithm
$\lambda = 10^{0}$	$2.779 \cdot 10^{-3}$	$1.243 \cdot 10^{-5}$
$\lambda = 10^{-2}$	$2.339 \cdot 10^{-3}$	$1.311 \cdot 10^{-5}$
$\lambda = 10^{-4}$	$1.673 \cdot 10^{-2}$	$5.069 \cdot 10^{-5}$
$\lambda = 10^{-6}$	$4.326 \cdot 10^{-2}$	$8.510 \cdot 10^{-5}$

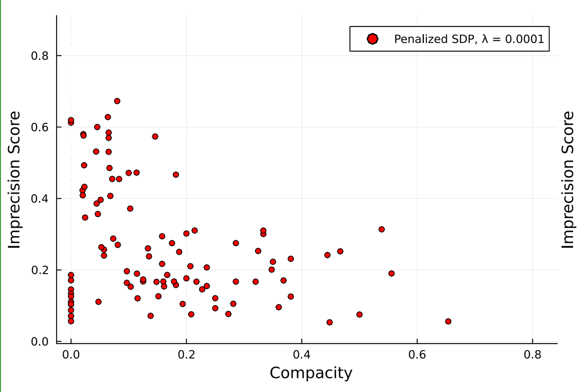
(Fig 8: *Average fractionnality with the separation procedure*)



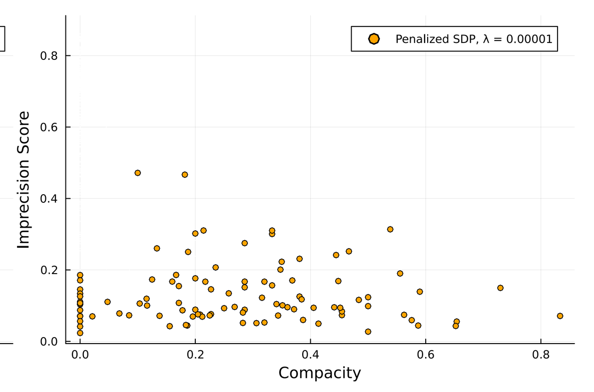
(Fig 3: $\lambda = 10^{-2}$)



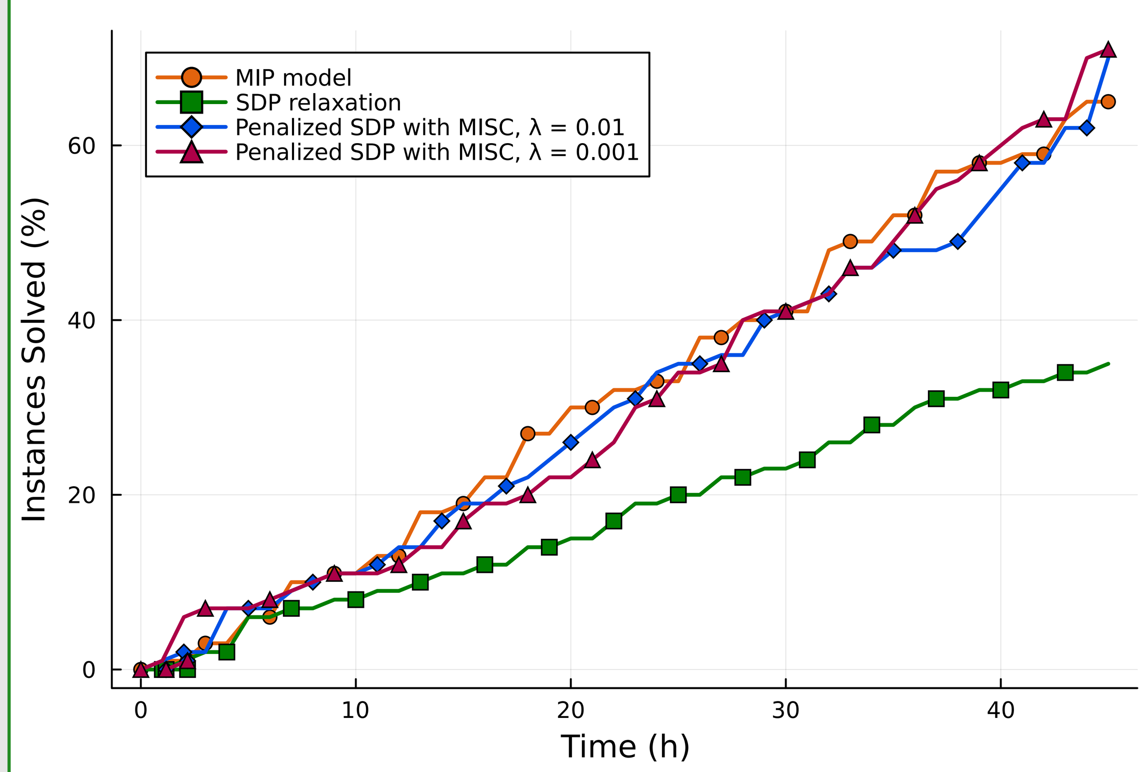
(Fig 4: $\lambda = 10^{-3}$)



(Fig 5: $\lambda = 10^{-4}$)



(Fig 6: $\lambda = 10^{-5}$)



(Fig 7: *Computation time for the studied models on the benchmark of 100 hard instances*)

References

[1] Cappello & Padilla (2025). **Bayesian variance change point detection with credible sets**. IEEE Trans. on Pattern Analysis and Machine Intelligence.

[2] Villuendas, Besançon & Malick (2025). **Knapsack with compactness: a semidefinite approach**. arXiv:2504.17543.

[3] Santini & Malaguti (2024). **The min-knapsack problem with compactness constraints and applications in statistics**. European Journal of Operational Research.