First-Order Methods for Distributionally Robust Mixed-Integer Optimization

Hubert Villuendas, Mathieu Besançon & Jérôme Malick
: hubert.villuendas@univ-grenoble.fr

University Grenoble Alpes, France

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Outline

1. Motivation and framework
   - Wasserstein distance
   - Ambiguity sets
2. Our approach
   - Entropic regularisation
   - Gradient estimators
   - Stochastic Frank-Wolfe algorithm
3. Numerical illustration
   - Quadratic minimum spanning tree

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From uncertainty to robustness

$$\underset{z \in \mathcal{Z}}{\text{minimize}} \quad f(z, \xi) \qquad\qquad \mathcal{Z} \subset \mathbf{R}^n \times \mathbf{Z}^{d-n}$$

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- Distributionally Robust Optimization (DRO):

$$\underset{z \in \mathcal{Z}}{\text{minimize}} \quad \sup_{\substack{\mathbf{Q} \in \text{Proba}(\Xi) \\ \mathbf{Q} \text{ close to } \widehat{\mathbf{P}}_N}} \mathbf{E}_{\zeta \sim \mathbf{Q}} \left[f(z, \zeta) \right]$$

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[*Esfahani and Kuhn, 2018*]

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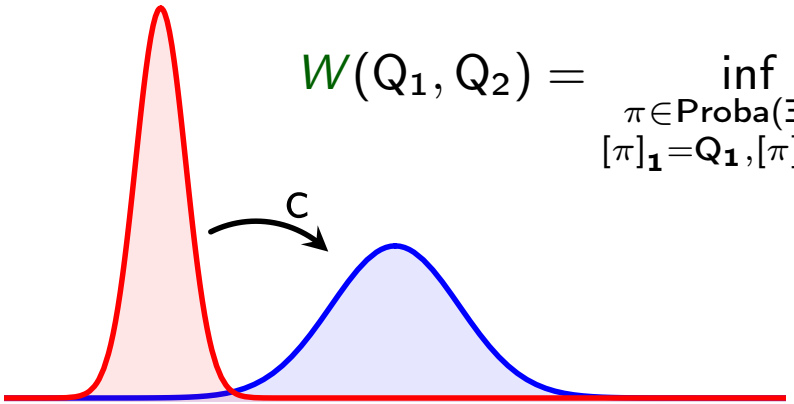

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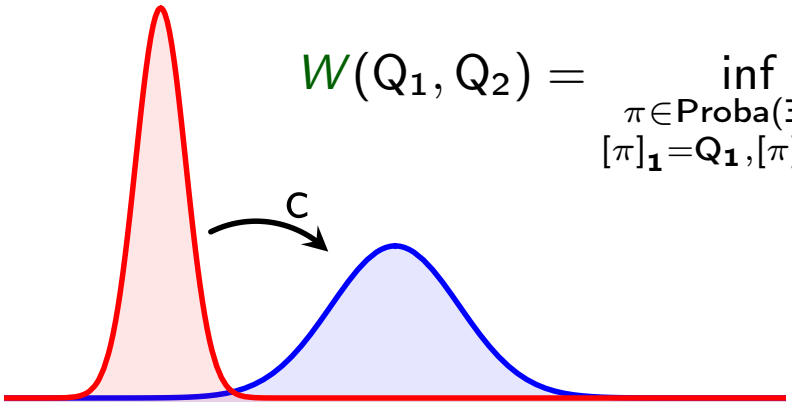

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- Chance-constrained vehicle routing **[*Ghosal and Wiesemann, 2020*]**

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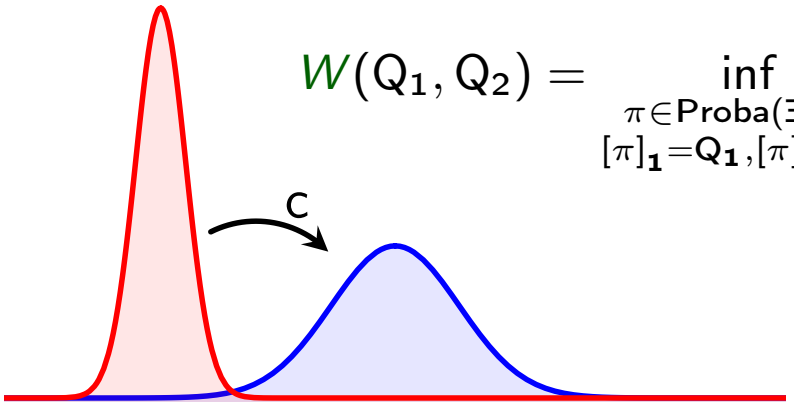

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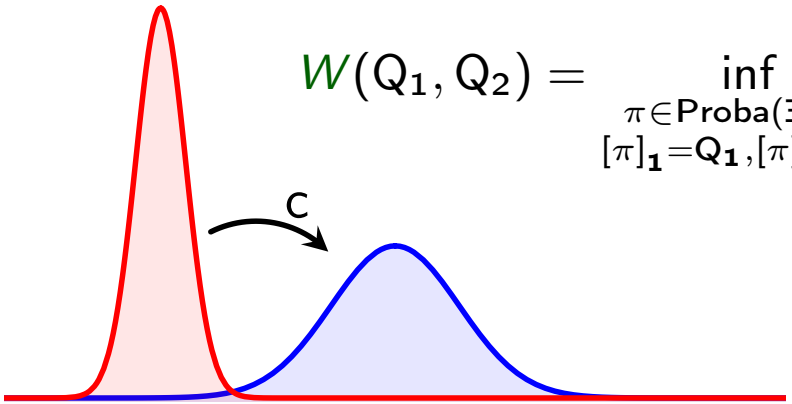

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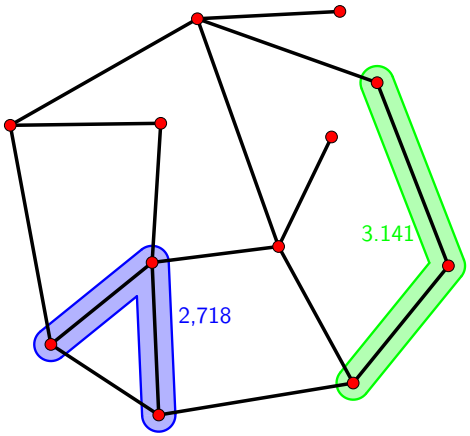
- Surgery allocation **[*Chow et al., 2022*]**
- Ressource allocation during a pandemic **[*Sun et al., 2023*]**

Running example — Quadratic Spanning Tree

$G = (V, E)$ on m edges, $\mathcal{C} \in \mathbf{R}_+^{m \times m}$ cost matrix.



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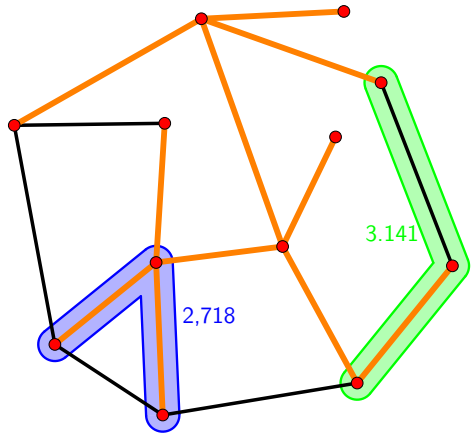

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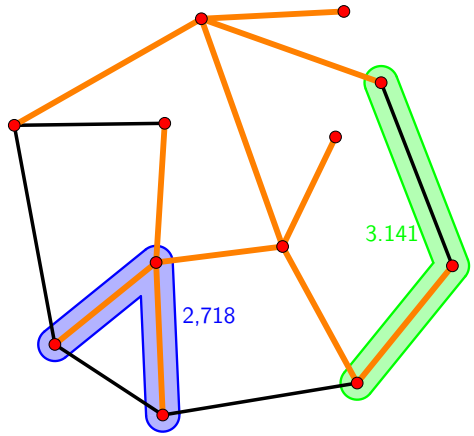

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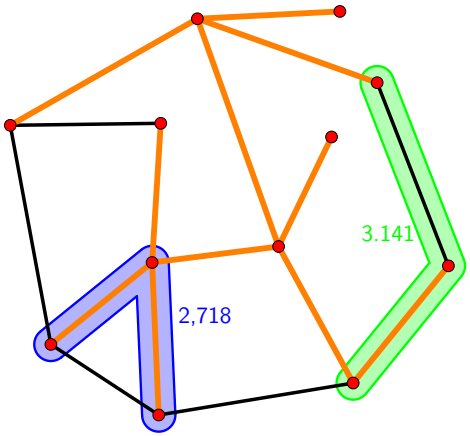

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- *Quadratic* minimum spanning tree problem is $\mathscr{NP}$-hard [***Assad and Xu, 1992***]

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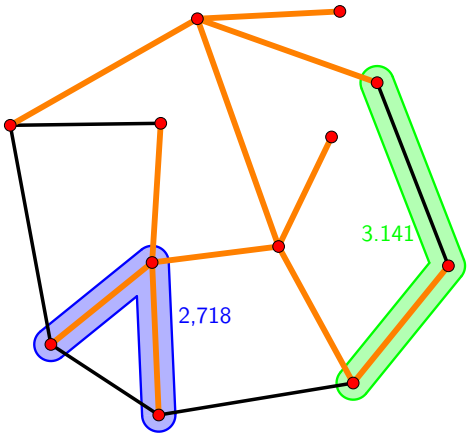

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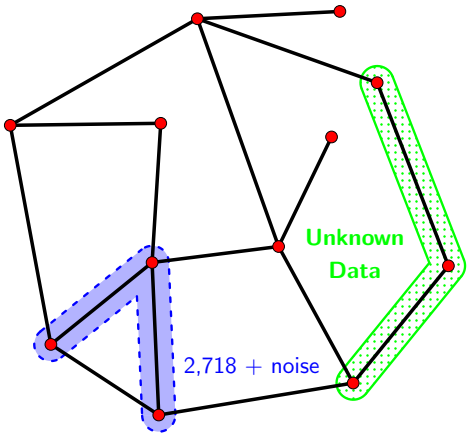

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Regularisation for WDRO

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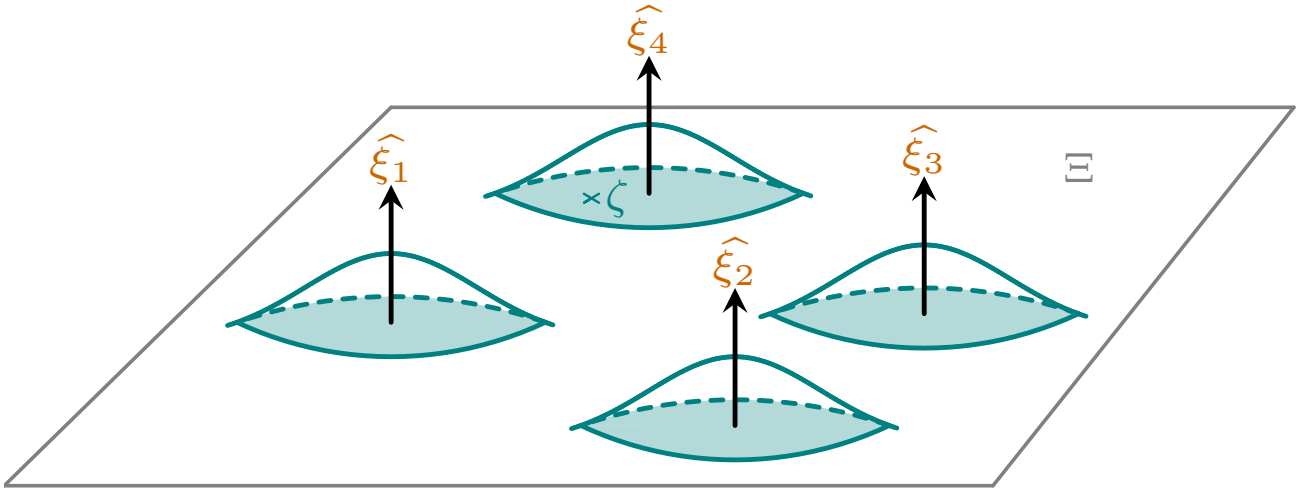

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Theorem (*Azizian et al., 2023*)

If f is convex and $f(\cdot,\zeta)$ is differentiable for all $\zeta \in \Xi$, then F is convex and differentiable, and:

$$\nabla_z F(z,\lambda) = \frac{1}{N}\sum_{k=1}^{N} \frac{\displaystyle\int_{\Xi} \nabla_z f(z,\lambda)\frac{h_{\widehat{\xi}}(z,\lambda;\zeta)}{(2\pi)^{d/2}\sigma}\exp\left(-\frac{\|\widehat{\xi}-\zeta\|^2}{2\sigma^2}\right)\mathrm{d}\zeta}{\displaystyle\int_{\Xi}\frac{h_{\widehat{\xi}}(z,\lambda;\zeta)}{(2\pi)^{d/2}\sigma}\exp\left(-\frac{\|\widehat{\xi}-\zeta\|^2}{2\sigma^2}\right)\mathrm{d}\zeta}$$

$$\frac{\partial}{\partial\lambda}F(z,\lambda) = \rho - \frac{1}{N}\sum_{k=1}^{N} \frac{\displaystyle\int_{\Xi} c\left(\widehat{\xi}_k,\zeta\right)\frac{h_{\widehat{\xi}}(z,\lambda;\zeta)}{(2\pi)^{d/2}\sigma}\exp\left(-\frac{\|\widehat{\xi}-\zeta\|^2}{2\sigma^2}\right)\mathrm{d}\zeta}{\displaystyle\int_{\Xi}\frac{h_{\widehat{\xi}}(z,\lambda;\zeta)}{(2\pi)^{d/2}\sigma}\exp\left(-\frac{\|\widehat{\xi}-\zeta\|^2}{2\sigma^2}\right)\mathrm{d}\zeta}$$

Gradient estimate

Samples : $\zeta_1^{(k)}, \ldots, \zeta_S^{(k)} \sim \mathcal{N}\left(\widehat{\xi}_k, \sigma^2 \mathbf{I}\right) \quad \forall k \in [\![N]\!]$ **[*Vincent et al., 2024*]**

$$\mathcal{G}_z(z, \lambda) \overset{\text{def}}{=} \frac{1}{N} \sum_{k=1}^{N} \frac{\frac{1}{S} \sum_{j=1}^{S} \nabla_z f\left(z, \zeta_j^{(k)}\right) h_{\widehat{\xi}}\left(z, \lambda ; \zeta_j^{(k)}\right)}{\frac{1}{S} \sum_{j=1}^{S} h_{\widehat{\xi}}\left(z, \lambda ; \zeta_j^{(k)}\right)}$$

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Example (Quadratic Spanning Tree — sampling cost)

$f(z, \xi) = z^\top \xi z$ and $\nabla_z f(z, \xi) = (\xi + \xi^\top) z$: easy.
Evaluation of $\mathcal{G}(z, \lambda)$: takes $S \times N$ samplings of $m \times m$ matrices: not easy.

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How to limit sampling costs?

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Proposition

Let $\mathcal{I}$: random subset. $\forall k \in \mathcal{I}$, let $\zeta_1^{(k)}, \dots, \zeta_S^{(k)} \sim \mathcal{N}\left(\widehat{\xi}_k, \sigma^2 \mathbf{I}\right)$ independant samples. Then $\mathbf{E}\left[\mathcal{G}^{\mathcal{I}}(z, \lambda)\right] \underset{S \to +\infty}{\longrightarrow} \nabla F(z, \lambda)$ almost surely.

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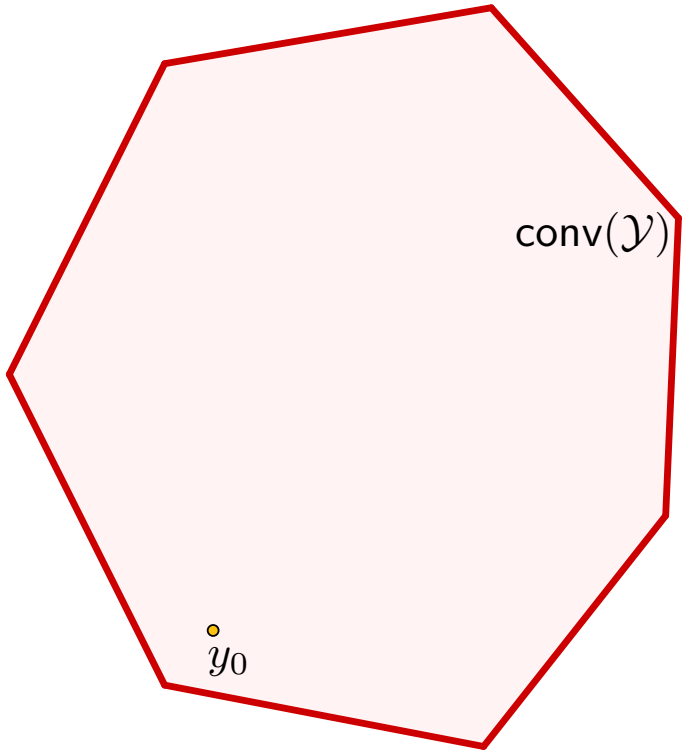

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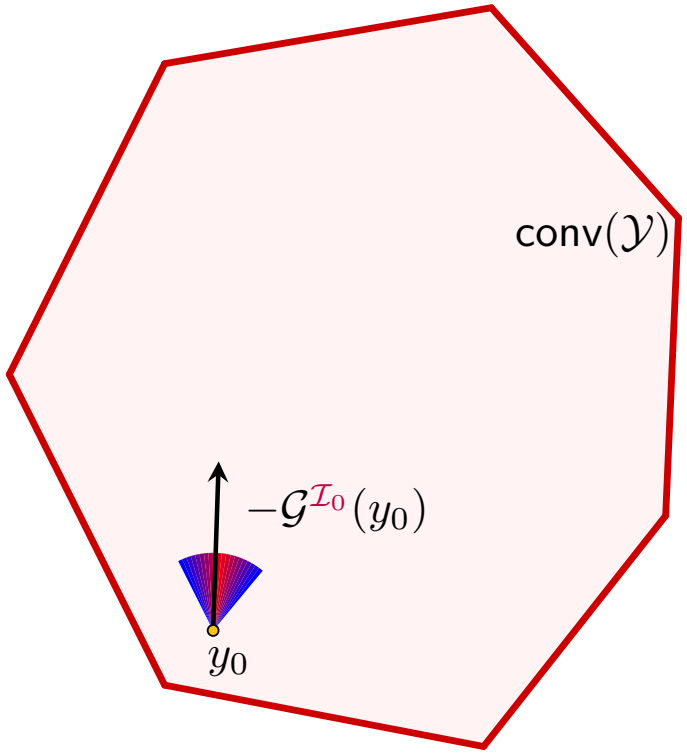

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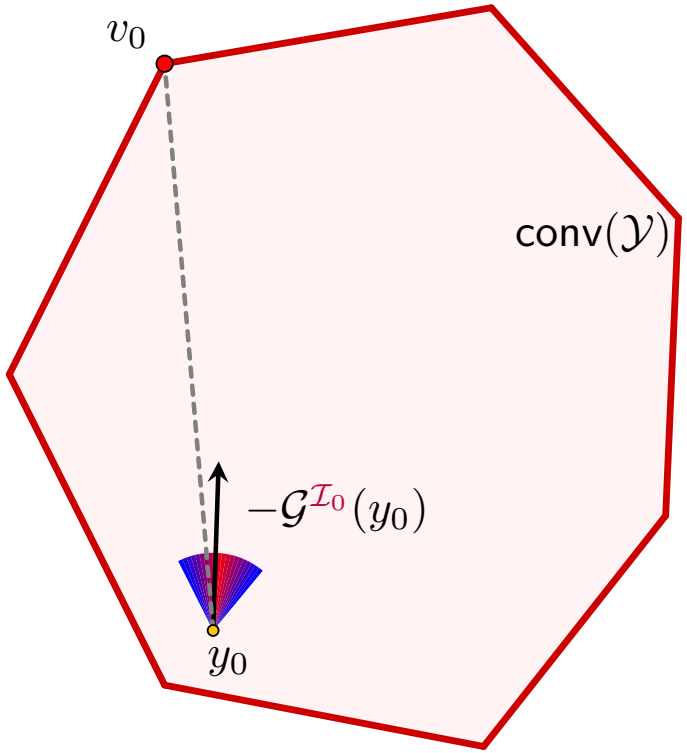

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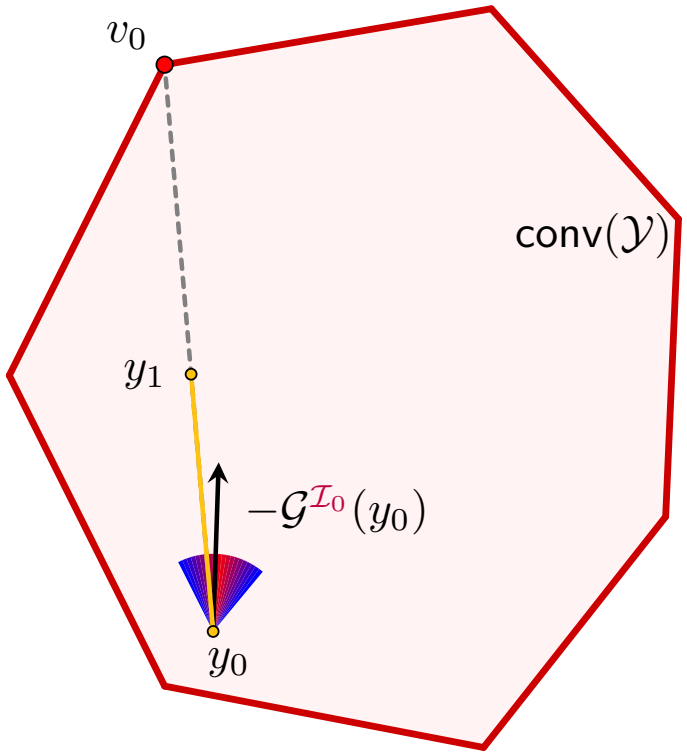

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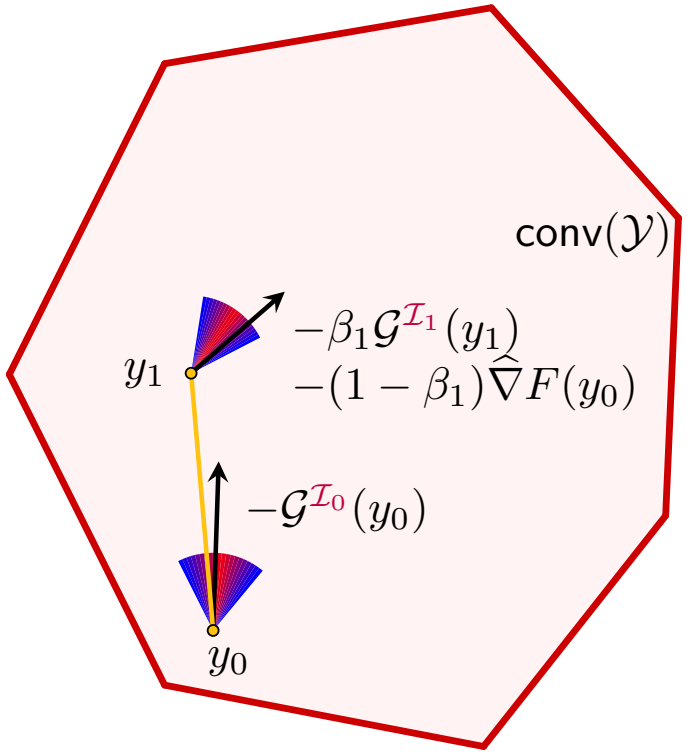

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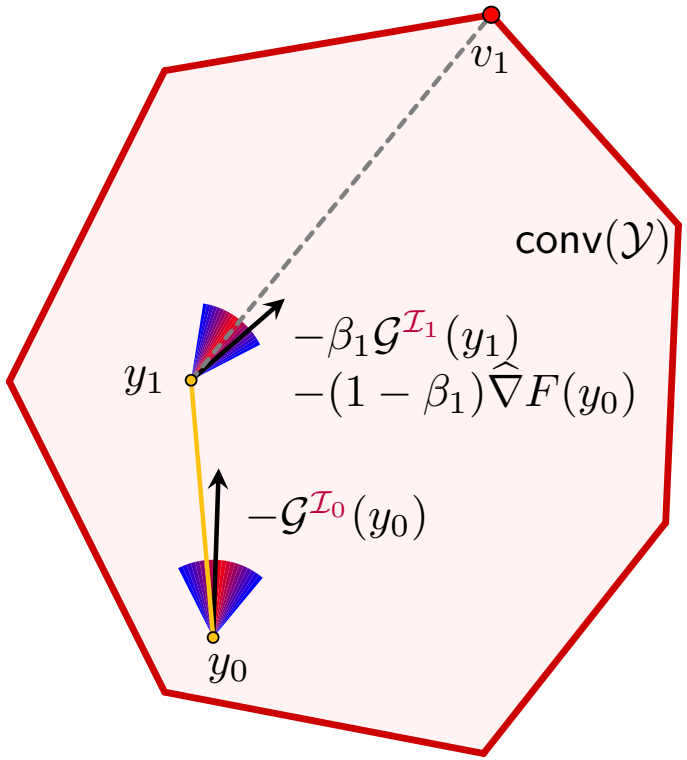

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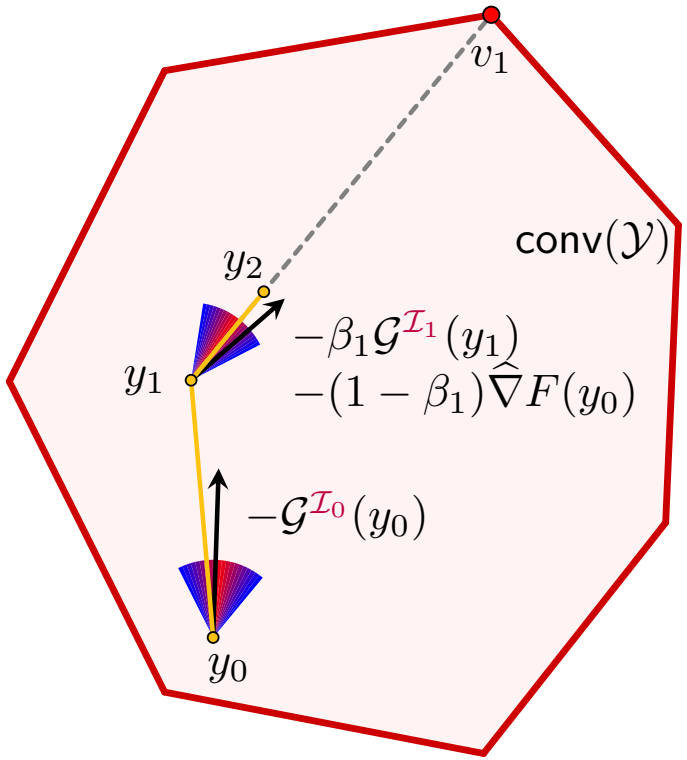

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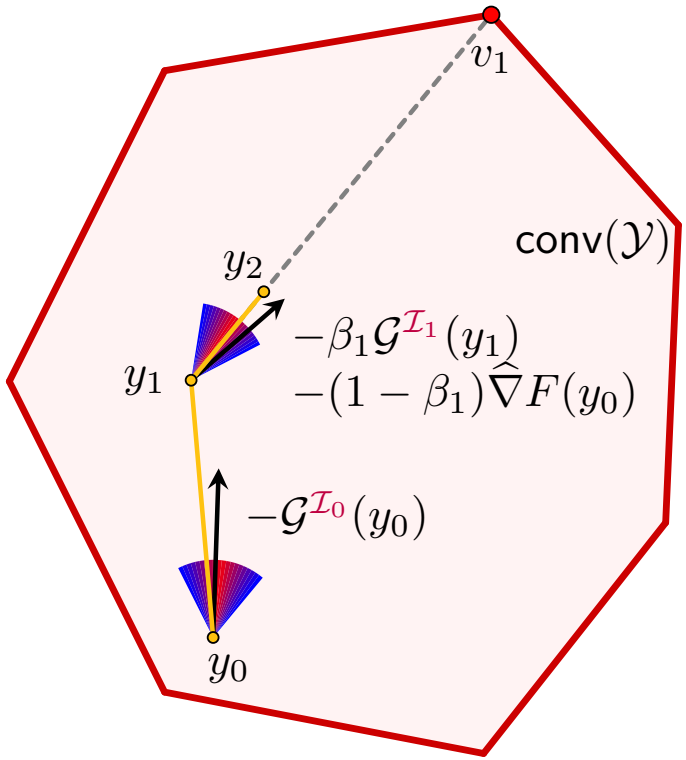

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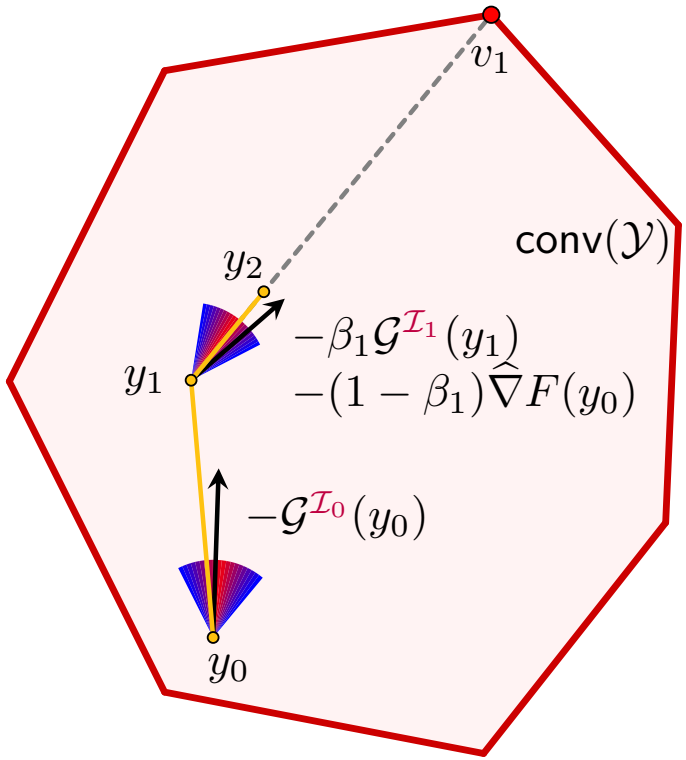

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Outline

1. Motivation and framework
   - Wasserstein distance
   - Ambiguity sets
2. Our approach
   - Entropic regularisation
   - Gradient estimators
   - Stochastic Frank-Wolfe algorithm
3. Numerical illustration
   - Quadratic minimum spanning tree

Experimental setup

	ERM	Robust
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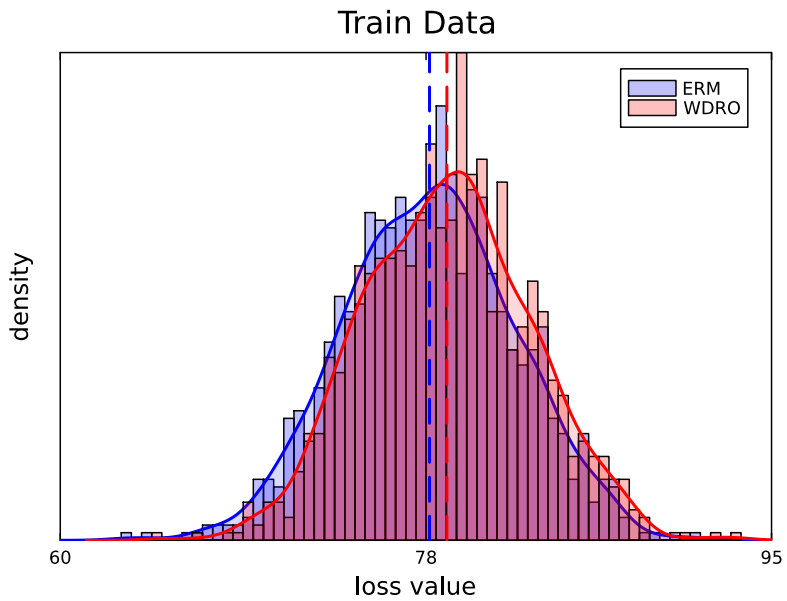
Experiments on Quadratic Minimum Spanning Tree



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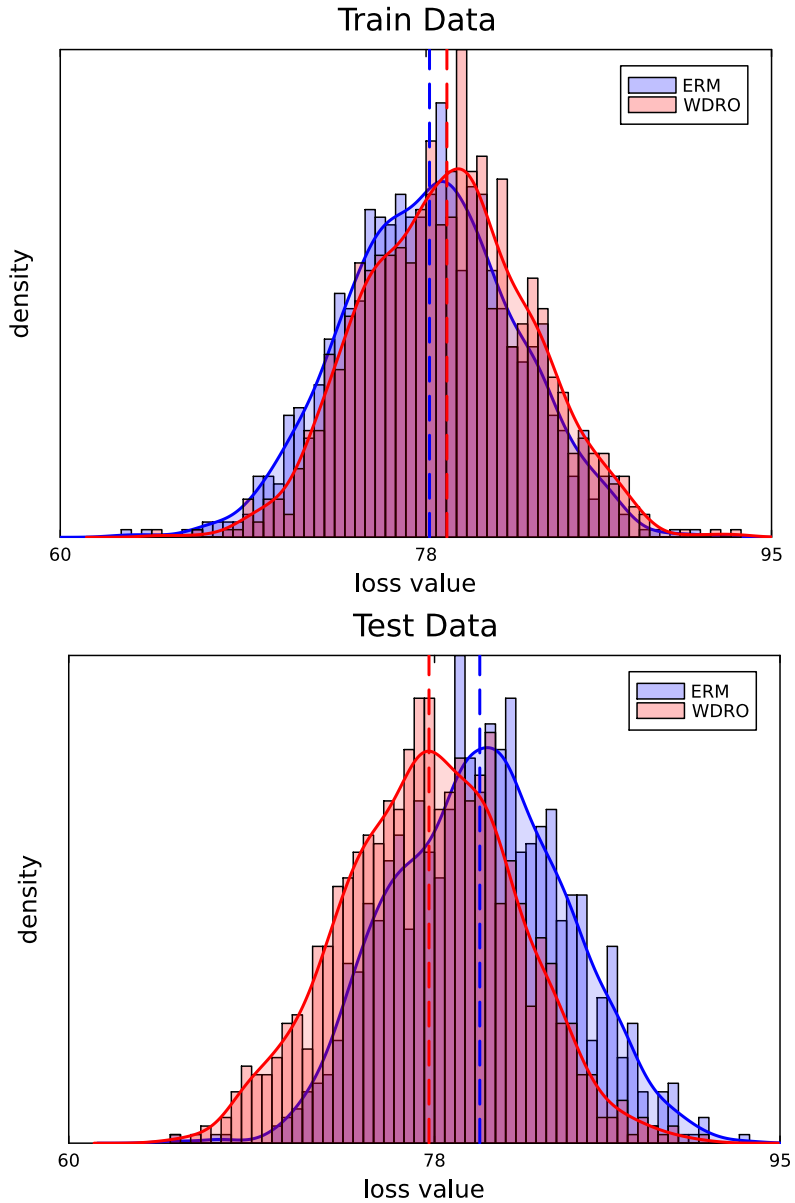

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