Mathematics for Computer Science

TD10

November 25th, 2026

Question. *Give a proof of the algorithm that generates uniformly a shortest path from A to B in a mesh.*

Let p be the number of horizontal moves and q the number of vertical moves required to go from A to B.
At each step, if p horizontal moves and q vertical moves remain, choose

$$H \quad \text{with probability} \quad \frac{p}{p+q},$$

and

$$V \quad \text{with probability} \quad \frac{q}{p+q}.$$

Then decrease the corresponding number of remaining moves by one.
We prove that this algorithm generates every shortest path with the same probability.
Consider a fixed shortest path P. It contains exactly p horizontal moves and q vertical moves.
The probability of generating P is the product of the probabilities of all successive choices made along this path.
Each time a horizontal move is selected, the numerator successively takes the values

$$p, p-1, \ldots, 1,$$

and each time a vertical move is selected, the numerator successively takes the values

$$q, q-1, \ldots, 1.$$

The denominators, independently of the order of the moves, are

$$p+q,\ p+q-1,\ \ldots,\ 1.$$

Therefore

$$\mathbb{P}(P) = \frac{p!\,q!}{(p+q)!}.$$

But

$$\binom{p+q}{p} = \frac{(p+q)!}{p!\,q!},$$

hence

$$\mathbb{P}(P) = \frac{1}{\binom{p+q}{p}}.$$

This probability does not depend on the chosen shortest path P.
Since there are exactly

$$\binom{p+q}{p}$$

shortest paths from A to B, the algorithm generates every shortest path with the same probability.
Thus the algorithm generates uniformly a shortest path from A to B.