Mathematics for Computer Science

TD6

October 14th, 2026

Question. *Prove, using a combinatorial argument, the recurrence equation:*

$$p_{n,k} = p_{n-1,k-1} + p_{n-k,k}.$$

Proof. Recall that for all $n \in \mathbb{N}$ and $k \leq n$, $p_{n,k}$ is the number of possible ways to dispach n balls in k *unlabelled* urn. In order to avoid symmetries, once we have disposed the balls in the urns, we sort urns by decreasing order of number of balls, and represent the situation by a *Ferrers diagramm*:

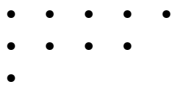


FIGURE 1: example with $n = 10$ and $k = 3$



FIGURE 2: another example with $n = 10$ and $k = 3$

Now, let us focus on a situation where n balls are dispached in k urns. We have two cases: either the last urn has exactly one ball (as in FIGURE 1); or the last urn has two or more balls (as in FIGURE 2). In particular, we thus have the following *disjoint union*:

$$\left\{ \begin{array}{c} \text{ways of dispaching} \\ n \text{ balls in } k \text{ urns} \end{array} \right\} = \left\{ \begin{array}{c} \text{cases where the last urn (after} \\ \text{sorting) has exactly 1 ball} \end{array} \right\} \sqcup \left\{ \begin{array}{c} \text{cases where the last urn} \\ \text{(after sorting) has} \geq 2 \text{ balls} \end{array} \right\} \tag{1}$$

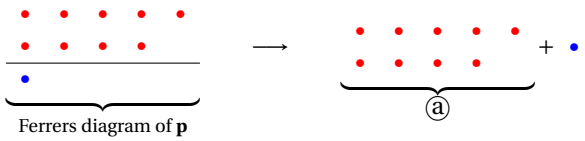
and we will count separately the number of elements of each of thoses sets.

To make the reading a bit easier, we will introduce the following notation:

Notation. *Let $n \in \mathbb{N}$ and $k \leq n$. We note $\mathcal{P}_{n,k}$ the set of (non-increasing) partitions of n in k terms, and in particular $\mathcal{P}^1_{n,k}$ (respectively $\mathcal{P}^{\geq 2}_{n,k}$) the subset of $\mathcal{P}_{n,k}$ in wich the smallest term is 1 (respectively ≥ 2).*

Now, suppose we have a partition $\mathbf{p}$ of n balls into k urns, *i.e.* $\mathbf{p} \in \mathcal{P}_{n,k}$. We distinguish two cases:

▷ Last urn has 1 ball: in that case, on the Ferrers diagram, we can always draw a line that separates the last urn from the other, as shown in this picture:



We then remark that the sub-diagram obtained by looking only what is above the line (all the red balls in the example above) is the Ferrers diagram of a partition of $n-1$ balls (since the last blue ball is the only one to have been set appart) into $k-1$ urns (since the line we drawn excludes the last urn — and only the last urn). Thus, there exists a unique partition $\mathbf{q}$ of $n-1$ balls into $k-1$ urns such that its Ferrers diagram is exactly ⓐ. We thus have an injective function

$$f : \left\{ \begin{array}{ccc} \mathcal{P}^1_{n,k} & \longrightarrow & \mathcal{P}_{n-1,k-1} \\ \mathbf{p} & \longmapsto & \mathbf{q} \end{array} \right.$$

Conversly, the reverse process takes any partition $\mathbf{q} \in \mathcal{P}_{n-1,k-1}$ and build a unique partition $\mathbf{p}$ of n into k terms by adding a $+1$ term – or, equivalently, by adding a single ball in a single urn – and such a partition is an element of $\mathcal{P}^1_{n,k}$. Thus, f is a bijection between $\mathcal{P}^1_{n,k}$ and $\mathcal{P}_{n-1,k-1}$, and those two sets have in particular the same cardinality. Since by definition, $\mathcal{P}_{n-1,k-1}$ has $p_{n-1,k-1}$ elements, we deduce that $\mathsf{Card}(\mathcal{P}^1_{n,k}) = p_{n-1,k-1}$.

▷ Last urn has ≥ 2 balls: in that case, on the Ferrers diagram, we can always draw a line that separates all first balls from each urns, as shown in this picture:



We then remark that the sub-diagram obtained by looking only what is on the right of the line (all the red balls in the example above) is the Ferrers diagram of a partition of $n-k$ balls (since we removed exactly k blue balls) into k urns. Thus, there exists a unique partition $\mathbf{q}$ of $n-k$ balls into k urns such that its Ferrers diagram is exactly ⓑ. We thus have an injective function

$$g : \left\{ \begin{array}{ccc} \mathcal{P}^{\geq 2}_{n,k} & \longrightarrow & \mathcal{P}_{n-k,k} \\ \mathbf{p} & \longmapsto & \mathbf{q} \end{array} \right.$$

Conversly, the reverse process takes any partition $\mathbf{q} \in \mathcal{P}_{n-k,k}$ and build a unique partition $\mathbf{p}$ of n into k terms by adding 1 to each term of the partition – or, equivalently, by adding a single ball to each existing urn – and such a partition is an element of $\mathcal{P}^{\geq 2}_{n,k}$. Thus, g is a bijection between $\mathcal{P}^{\geq 2}_{n,k}$ and $\mathcal{P}_{n-k,k}$, and those two sets have in particular the same cardinality. Since by definition, $\mathcal{P}_{n-k,k}$ has $p_{n-k,k}$ elements, we deduce that $\mathsf{Card}(\mathcal{P}^{\geq 2}_{n,k}) = p_{n-k,k}$.

We can now count the total amount of partitions of n in k terms. Since the union of Equation (**??**) is disjoint, and that we calculated the cardinality of each sets above, we have the wanted equality:

$$\begin{aligned} \mathcal{P}_{n,k} &= \mathcal{P}^{1}_{n,k} \sqcup \mathcal{P}^{\geq 2}_{n,k} \\ \mathsf{Card}(\mathcal{P}_{n,k}) &= \mathsf{Card}(\mathcal{P}^{1}_{n,k}) + \mathsf{Card}(\mathcal{P}^{\geq 2}_{n,k}) \\ p_{n,k} &= p_{n-1,k-1} + p_{n-k,k}. \end{aligned}$$

□