Mathematics for Computer Science

TD7

October 21th, 2026

Question. *Prove that the amount of water in each jug is always a multiple of* GCD(a, b).

Proof. Let us write $d = \text{GCD}(a, b)$ to make the redaction shorter and clearer. We prove that, at every step of the process, the amount of water contained in each jug is a multiple of d. We thus propose a proof by induction.

- ▷ Step 0: initially, both jugs are empty, so their contents are 0, which is a multiple of d.

Suppose that we are currently at step k, for $k \geq 0$. The two jugs contain some quantity of water, lets say x and y, that we assume are both multiple of d. In particular, GCD(x, y) is itself a multiple of d.

- ▷ Step $k \to k+1$: we check that every possible move preserves the property. Going from step k to step $k+1$ imposes to make one of the following actions:

  1. Emptying a jug replaces x or y's value by 0 which is a multiple of d. The untouched jug keeps the same amount of water which is a multiple of d by our induction hypothesis.
  2. Filling a jug completely replaces x (respectively y) by a (respectively b) which is a multiple of d. The untouched jug keeps the same amount of water wich is a multiple of d by our induction hypothesis.
  3. Pouring water from one jug into the other continues until either the first jug is empty or the second one is full. Without loss of generality, we suppose that we pour from the a-jug into the b-jug. In that case, the transferred quantity is
  $$q = \min(x, b - y).$$
  Since x, b and y are multiples of d, $b - y$ is a multiple of d, and finally q is also a multiple of d. Now, the new contents of each jug is $x - q$ and $y + q$ respectively, which are thus both multiples of d by induction hypothesis.

  Every moves preserves the property that the quantity is a multiple of d, thus we we reach step $k+1$, both jugs contains an amount of water which is a multiple of d.

Therefore, that the amount of water in each jug is always a multiple of GCD(a, b). □