Mathematics for Computer Science

TD8

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Question. *Prove Euler's formula for connected planar graphs. You can use the subcase for trees as a lemma.*

We first formalize the subcase for tree in the form of the following lemma:

Lemma 1. *Let $\mathcal{T}$ be a tree on n vertices and m edges. Then $m = n - 1$.*

For the sake of completness, we provide a proof for this lemma. As a small remark: this fact is probably very well known from every computer scientist, and is an immediate corollary of Euler's formula for planar graphs. Since the goal of this excercise session is to use **Lemma 1** to provide a full proof of Euler's formula, we can't prove the lemma itself using Euler's formula in order to prevent a circular argument.

Proof. We prove **Lemma 1** (*without Euler's formula!*) by induction on n the number of vertices of the tree.

- ▷ *Base case* : let us consider $\mathcal{T}$ a tree with only $n = 1$ vertex. Then, since there is no possibility to join a vertex to itself, $\mathcal{T}$ can't have any edge. Therefore $m = 0 = 1 - 1$, and the formula holds for a tree on 1 vertex.
- ▷ *Induction step* : let $n \geq 2$ be such that any tree on $n-1$ vertices verifies the statement of **Lemma 1**. Let $\mathcal{T} = (V, E)$ be a tree on n vertices, and let m be its number of edges. Take $v \in V$ a leaf of the tree, and call $e \in E$ the single edge connected to v. Then the subgraph $\mathcal{T}' = (V \setminus \{v\}, E \setminus \{e\})$ is also a tree, that has exactly $n-1$ vertices and $m-1$ edges. By induction hypothesis, we thus have that $m - 1 = n - 1 - 1$. Therefore, $m = n - 1$, and the formula holds for any tree of size n.
- ▷ *Conclusion* : by the induction principle, for all $n \geq 1$, any tree on n vertices has exactly $m = n - 1$ edges.

□

Since we showed **Lemma 1** without Euler's formula, we can apply this lemma to prove the wanted result.

Proof. Let $G = (V, E)$ be a connected planar graph on n vertices. We will prove Euler's formula by induction on $|F|$ the number of faces delimited by a planar representation of G.

- ▷ *Base case* : if $|F| = 1$, it means that G can't contain any cycle, which implies that G is a tree. We can thus apply **Lemma 1** and state that $|E| = n - 1$. We therefore have $n - |E| + |F| = n - n + 1 + 1 = 2$, and Euler's formula is verified for this base case.
- ▷ *Induction step* : suppose that $|F| \geq 2$ is such that any connected planar graph $G' = (V', E')$ with $|F| - 1$ faces verifies Euler's formula : $|V'| - |E'| + |F| - 1 = 2$. Since we consider the case where G has $|F| \geq 2$ faces, G contains at least a cycle $\mathcal{C} = (V_{\mathcal{C}}, E_{\mathcal{C}})$. Consider $e \in E_{\mathcal{C}}$ an edge of G contained in this cycle. Let us build the subgraph G' obtained by removing edge e : $G' = (V, E \setminus \{e\})$. Since the two endpoints of e are still connected to other edges of the cycle $\mathcal{C}$, the resulting graph G' remains connected. Removing an edge from a planar graph leaves a planar graph, therefore G' is a connected planar graph. Moreover, removing e merges the faces adjacent to e into a single face, which implies that G' has exactly $|F| - 1$ faces. By the induction hypothesis on graph G':

$$\begin{aligned} |V| - |E \setminus \{e\}| + |F| - 1 &= 2 \\ n - (|E| - 1) + |F| - 1 &= 2 \\ n - |E| + |F| &= 2 \end{aligned}$$

  and thusn graph G verifies Euler's formula.
- ▷ *Conclusion* : by the induction principle, for any number of faces $|F| \geq 1$, any connected planar graph with $|F|$ faces verifies Euler's formula, thus proving the wanted result.

□