# Mathematics for Computer Science

TD9

November 18th, 2026

**Question.** *Prove that any outer-planar graph is* 3*-colorable.*

We first prove the lemma from the TD:

**Lemma 1.** *Let $G = (V, E)$ be an $n$-vertex outer-planar graph, with $n \geq 3$. Then the following holds:*

(i) *If $G$ is maximal* (i.e. *no extra edge* uv *can be added to $E$ without breaking the property of outer-planarity, with* $u, v \in V$*) then at least one of $G$'s vertices has degree 2.*

(ii) *For any outer-planar $G$ (not necessarily maximal), at least one of $G$'s vertices has degree less than* 2.

*Proof.* We prove **Lemma 1**:

(i) Let $G$ maximal outer-planar graph. We will first show that every internal face of $G$ is a triangle. Indeed, if that hadn't been the case, it would have been prossible to find $v_1, v_2, v_3, \ldots, v_r \in V$ ($r \geq 4$) such that the polygon $\mathcal{P} = v_1 \ldots v_r$ is an internal face of $G$. But then, we note that the edge $v_1 v_2$ is a diagonal of $\mathcal{P}$ that is not present in $E$ and that is never cutting through any existing edge of $G$, so we could add the edge $v_1 v_3$ to $E$ without breaking the property of outer-planarity, thus contradicting the maximality of $G$. So, $G$ is an arrangement of non-overlaping triangles.

Now, we want to show that $G$ must have a vertex with degree exactly 2. To understand how this work, we seek to answer to the question: how can we construct a triangulated graph with all vertices on a circle?

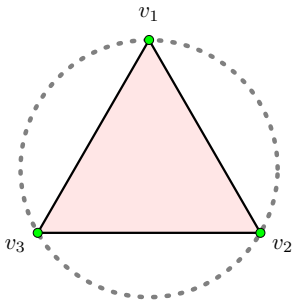


FIGURE A: smallest maximal outer-planar graph possible: 3 points, all vertices have degree 2. ✔

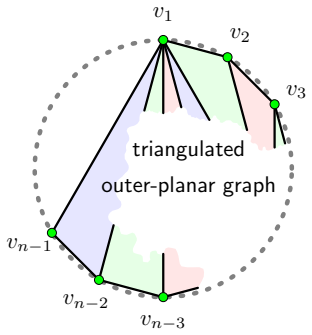


FIGURE B: a maximal outer-planar graph on $n-1$ vertices. No particular assumption on the degree.

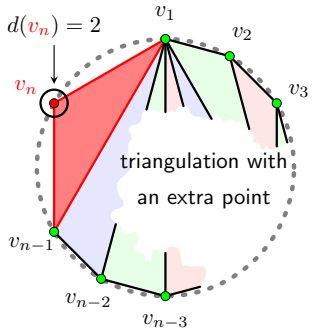


FIGURE C: we add the $n^{\text{th}}$ vertex: for $G$ to be maximal, $v_n$ needs to connect exactly to its two neighbors: $d(v_n) = 2$. ✔

FIGURE A shows that the smallest maximal outer-planar graph, with 3 vertices, verifies the property of having at least a vertex with degree exactly 2. We want to build our graph $G$ by iteratively adding vertices, and see if the property is verified at each step. FIGURE B illustrates the situation we have just before adding a new vertex: $n-1$ points are dispached on a circle. Note that the labeling is here to help us counting how many points are displayed, and does not indicate a specific ordering of the points. If we want to add an extra vertex "$v_n$", we could theorically put this new point anywhere on the circle, in between any two points. Without loss of generality, we place our $(n+1)^{\text{th}}$ point between points labeled "$n$" and "1".

FIGURE C illustrates what happens when the $n^{\text{th}}$ vertex is added: since we want to keep the property of maximality, $v_n$ needs to connect to its two neighbors on the circle, on the example, $v_1$ and $v_{n-1}$. It thus creates a triangle, displayed in red on the figure. Since $v_n$ is not connected to any other vertex than the two other vertex of the triangle, $d(v_n) = 2$.

This reasonning shows that, at each step of the construction, we can find at least a vertex with degree exacty 2, thus establishing the wanted result.

(ii) We note that this second statement is a direct application of the first property.

Indeed, if $G = (V, E)$ is an outer-planar graph that is not maximal, it means that we can find two vertices $u_1, v_1 \in V$ such that the graph $G_1 = (V, E \cup \{u_1v_1\})$ is outer-planar. If $G_1$ is not outer-planar, we find two vertices $u_2, v_2 \in V$ such that the graph $G_2 = (V, E \cup \{u_2v_2\})$ is outer-planar; and so on. By repeating this process until the graph is maximal, we can iteratively build a sequence of outer-planar graphs $G_1, G_2, \ldots, G_{\max}$ with vertex set $V$ that are all outer-planar, and $G_{\max}$ is maximal.

Applying the first property to $G_{\max}$ gives that there exists $v \in V(G_{\max})$ such that $v$ has degree exactly 2 in $G_{\max}$. Since $G$ and $G_{\max}$ have identical vertex sets and that $G$ is a sub-graph of $G_{\max}$, we have that $v \in V$, and the number of edges connected to $v$ in $G$ is less or equal to the number of edges connected to $v$ in $G_{\max}$, or in other terms, $d_G(v) \leq d_{G_{\max}}(v) = 2$.

□

We can now turn ourselves to the proof of the 3-colorability of every outer-planar graph:

*Proof.* Let $G = (V, E)$ be an outer-planar graph. We prove by induction on $n = |V|$ that $G$ is 3-colorable.

▷ *Base case* : for $n \leq 3$, the result is immediate: just assign a unique color to every vertex to get a 3-coloring of the graph.

▷ *Induction step* : now suppose that every outer-planar graph with $n-1$ vertices is 3-colorable, and let $G$ be an outer-planar graph with $n$ vertices. By **Lemma 1**, point (ii), $G$ contains a vertex $v$ of degree at most 2. Suppose that $d(v)$ has degree 2, and let us call $u_1$ and $u_2$ the two (distincts) neighbors of $v$.

From this, build graph $G' = (V \setminus \{v\}, E \setminus \{u_1v, u_2v\})$, the graph obtained by removing $v$ and all edges incident to $v$. Since removing a vertex preserves the property of outer-planarity, $G'$ is still outer-planar, and by the induction hypothesis, $G'$ admits a proper coloring with three colors. Lets choose colors red, green and blue, and denote by $R$, $G$ and $B$ the sets of respective color red, green and blue.

$$V \setminus \{v\} = R \sqcup G \sqcup B$$

Now, by the pigeon-hole principle, there exists at least one set from $\{R, G, B\}$ such that contains none of $u_1, u_2$. We can thus assign its associated color to $v$. The resulting coloring is a proper 3-coloring of $G$.

If $d(v) = 1$, the reasoning is the same: color $v$ with one of the two unused color by the neighbor of $v$.

▷ *Conclusion* : hence, by induction, every outer-planar graph is 3-colorable.

□